

HARNACK INEQUALITY FOR NONDIVERGENT PARABOLIC OPERATORS ON RIEMANNIAN MANIFOLDS

SEICK KIM, SOOJUNG KIM, AND KI-AHM LEE

ABSTRACT. We consider second-order linear parabolic operators in non-divergence form that are intrinsically defined on Riemannian manifolds. In the elliptic case, Cabré proved a global Krylov-Safonov Harnack inequality under the assumption that the sectional curvature of the underlying manifold is nonnegative. Later, Kim improved Cabré's result by replacing the curvature condition by a certain condition on the distance function. Assuming essentially the same condition introduced by Kim, we establish Krylov-Safonov Harnack inequality for nonnegative solutions of the non-divergent parabolic equation. This, in particular, gives a new proof for Li-Yau Harnack inequality for positive solutions to the heat equation in a manifold with nonnegative Ricci curvature.

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1. INTRODUCTION AND MAIN RESULTS

In this paper, we study Harnack inequalities for solutions of second-order parabolic equations of non-divergence type on Riemannian manifolds. Let (M, g) be a smooth, complete Riemannian manifold of dimension n . For $x \in M$ and $t \in \mathbb{R}$, let $A_{x,t}$ be a positive definite symmetric endomorphism of $T_x M$, where $T_x M$ is the tangent space of M at x . We denote $\langle X, Y \rangle := g(X, Y)$ and $|X|^2 := \langle X, X \rangle$ and assume that

$$(1) \quad \lambda |X|^2 \leq \langle A_{x,t} X, X \rangle \leq \Lambda |X|^2, \quad \forall (x, t) \in M \times \mathbb{R}, \quad \forall X \in T_x M$$

for some positive constants λ and Λ . We consider a second-order, linear, uniformly parabolic operator \mathcal{L} defined by

$$(2) \quad \mathcal{L}u = Lu - u_t := \text{trace}(A_{x,t} \circ D^2 u) - u_t \quad \text{in } M \times \mathbb{R},$$

where \circ denotes composition of endomorphisms and $D^2 u$ denotes the Hessian of the function u defined by

$$D^2 u \cdot X = \nabla_X \nabla u,$$

where $\nabla u(x) \in T_x M$ is the gradient of u at x . Notice that in the special case when $A_{x,t} \equiv \text{Id}$, the equation $\mathcal{L}u = 0$ simply becomes the usual heat equation $u_t - \Delta u = 0$.

In the elliptic setting, Cabré proved in a remarkable paper [Ca] that if the underlying manifold M has nonnegative sectional curvature, then Krylov-Safonov type (elliptic) Harnack inequality holds for solutions of uniformly elliptic equations in non-divergence form. Later, Kim [K] improved Cabré's result by removing the sectional curvature assumption and imposing a certain condition on distance function which, in the parabolic setting, should read as follows: For all $p \in M$, we have

$$(3) \quad \Delta d_p(x) \leq \frac{n-1}{d_p(x)} \quad \text{for } x \notin \text{Cut}(p) \cup \{p\},$$

$$(4) \quad Ld_p(x) \leq \frac{a_L}{d_p(x)} \quad \text{for } x \notin \text{Cut}(p) \cup \{p\}, \quad t \in \mathbb{R},$$

where $d_p(x) = d(p, x)$ is the geodesic distance between p and x , $\text{Cut}(p)$ denotes the cut locus of p , and a_L is some positive constant that is fixed by the operator L . We shall prove that if the above conditions (3) and (4) hold, then we have Krylov-Safonov Harnack inequality for the parabolic operator \mathcal{L} ; i.e., if u is a (smooth) nonnegative solution of $\mathcal{L}u = f$ in a cylinder $K_{2R} := B_{2R}(x_0) \times (t_0 - 4R^2, t_0)$, where $x_0 \in M$ and $t_0 \in \mathbb{R}$, then we have

$$(5) \quad \sup_{K_R^-} u \leq C \left\{ \inf_{K_R^+} u + R^2 \left(\frac{1}{\text{Vol}(K_{2R})} \int_{K_{2R}} |f|^{n+1} \right)^{\frac{1}{n+1}} \right\},$$

where $K_R^- := B_R(x_0) \times (t_0 - 3R^2, t_0 - 2R^2)$, $K_R^+ := B_R(x_0) \times (t_0 - R^2, t_0)$, Vol denotes the volume, and C is a uniform constant depending only on n, λ, Λ and a_L . It is well known that the condition (3) holds if the manifold M has nonnegative Ricci curvature. Also, as it is proved in [K], the condition (4) is satisfied, for example, if for all $x \in M$ and any unit vector $e \in T_x M$, we have $\mathcal{M}^-[R(e)] \geq 0$. Here, $R(e)$ is the Ricci transformation of $T_x M$ into itself given by $R(e)X := R(X, e)e$, where $R(X, Y)Z$ is the Riemannian curvature tensor, and

$$\mathcal{M}^-[R(e)] = \mathcal{M}^-[R(e), \lambda, \Lambda] := \lambda \sum_{\kappa_i > 0} \kappa_i + \Lambda \sum_{\kappa_i < 0} \kappa_i,$$

where κ_i are eigenvalues of the (symmetric) endomorphism $R(e)$. In the case when \mathcal{L} is the heat operator and M has nonnegative Ricci curvature, then the condition $\mathcal{M}^-[R(e)] \geq 0$

is satisfied and thus the Harnack inequality (5) holds; i.e., if M has nonnegative Ricci curvature, then we have

$$\sup_{K_R^-} u \leq C_n \left\{ \inf_{K_R^+} u + R^2 \left(\frac{1}{\text{Vol}(K_{2R})} \int_{K_{2R}} |u_t - \Delta u|^{n+1} \right)^{\frac{1}{n+1}} \right\},$$

where C_n is a constant that depends only on the dimension n . This, in particular implies the Harnack inequality of Li and Yau [LY]. Also, in the case when M has nonnegative sectional curvature, then the condition $\mathcal{M}^-[R(e)] \geq 0$ is trivially satisfied and we have the inequality (5) with a constant C depending only on n, λ, Λ , which especially reproduces the Harnack inequality by Krylov and Safonov [KS] in the Euclidean space.

One crucial ingredient in proving the Euclidean Krylov-Safonov Harnack inequality is the Krylov-Tso estimate, which is the parabolic counterpart of the Aleksandrov-Bakelman-Pucci (ABP) estimate. The Krylov-Tso estimate as well as the classical ABP estimate is proved using affine functions, which have no intrinsic interpretation in general Riemannian manifolds. In the elliptic case, Cabré ingeniously overcame this difficulty by replacing the affine functions by quadratic functions; quadratic functions have geometric meaning as the square of distance functions. Following Cabré's approach, we introduce an intrinsically geometric version of Krylov-Tso normal map, namely,

$$\Phi(x, t) := \left(\exp_x \nabla_x u(x, t), -\frac{1}{2} d(x, \exp_x \nabla u(x, t))^2 - u(x, t) \right).$$

The map Φ is called the parabolic normal map related to $u(x, t)$. A few remarks are in order regarding the normal map. In the classical ABP (and Krylov-Tso) estimate, an affine function concerning with the (elliptic) normal map $x \mapsto \nabla u(x)$ plays a role to bound the maximum of u by estimating the measure of the image of the normal map. Since an affine function cannot be generalized naturally to an intrinsic object in Riemannian manifolds, Cabré used paraboloids instead in [Ca]. The map

$$p \mapsto \min_{\Omega} \{u(x) - p \cdot x\} \quad \text{for a domain } \Omega$$

is considered (up to a sign) as the Legendre transform of u . Krylov [Kr] discovered the parabolic version of the Aleksandrov-Bakelman maximum principle and Tso [T] later simplified his proof by using the map

$$(x, t) \mapsto (\nabla_x u(x, t), \nabla_x u(x, t) \cdot x - u(x, t)).$$

We end the introduction by stating our main theorems. The rest of the paper shall be devoted to their proof. Below and hereafter, we denote

$$\oint_Q f := \frac{1}{\text{Vol}(Q)} \int_Q f$$

and

$$K_r(x_0, t_0) := B_r(x_0) \times (t_0 - r^2, t_0], \quad (x_0, t_0) \in M \times \mathbb{R}.$$

Theorem 1.1 (Harnack inequality). *Suppose conditions (3), (4) hold. Let u be a nonnegative smooth function in $K_{2R}(x_0, 4R^2)$, where $x_0 \in M$ and $R > 0$. Then, we have*

$$(6) \quad \sup_{K_R(x_0, 2R^2)} u \leq C \left\{ \inf_{K_R(x_0, 4R^2)} u + R^2 \left(\oint_{K_{2R}(x_0, 4R^2)} |\mathcal{L}u|^{n+1} \right)^{\frac{1}{n+1}} \right\},$$

where C is a uniform constant depending only on n, λ, Λ and a_L .

Theorem 1.2 (Weak Harnack inequality). *Suppose the conditions (3), (4) hold. Let u be a nonnegative smooth function satisfying $\mathcal{L}u \leq f$ in $K_{2R}(x_0, 4R^2)$, where $x_0 \in M$ and $R > 0$. Then, we have*

$$\left(\int_{K_R(x_0, 2R^2)} u^p \right)^{\frac{1}{p}} \leq C \left\{ \inf_{K_R(x_0, 4R^2)} u + R^2 \left(\int_{K_{2R}(x_0, 4R^2)} |f^+|^{n+1} \right)^{\frac{1}{n+1}} \right\}; \quad f^+ := \max(f, 0),$$

where $p \in (0, 1)$ and C are uniform constants depending only on n, λ, Λ and a_L .

2. PRELIMINARIES

Let (M, g) be a smooth, complete Riemannian manifold of dimension n , where g is the Riemannian metric and $\text{Vol} := \text{Vol}_g$ is the reference measure on M . We denote $\langle X, Y \rangle := g(X, Y)$ and $|X|^2 := \langle X, X \rangle$ for $X, Y \in T_x M$, where $T_x M$ is the tangent space at $x \in M$. Let $d(\cdot, \cdot)$ be the distance function on M . For a given point $y \in M$, $d_y(x)$ denotes the distance function from y , i.e., $d_y(x) := d(x, y)$.

We recall the exponential map $\exp : TM \rightarrow M$. If $\gamma_{x,X} : \mathbb{R} \rightarrow M$ is the geodesic starting from $x \in M$ with velocity $X \in T_x M$, then the exponential map is defined by

$$\exp_x(X) := \gamma_{x,X}(1).$$

We note that the geodesic $\gamma_{x,X}$ is defined for all time since M is complete. Given two points $x, y \in M$, there exists a unique minimizing geodesic $\exp_x(tX)$ joining x to y with $y = \exp_x(X)$ and we will write $X = \exp_x^{-1}(y)$.

For $X \in T_x M$ with $|X| = 1$, we define

$$t_c(X) := \sup \{ t > 0 : \exp_x(sX) \text{ is minimizing between } x \text{ and } \exp_x(tX) \}.$$

If $t_c(X) < +\infty$, $\exp_x(t_c(X)X)$ is a cut point of x . The cut locus of x is defined as the set of all cut points of x , that is,

$$\text{Cut}(x) := \{ \exp_x(t_c(X)X) : X \in T_x M \text{ with } |X| = 1, t_c(X) < +\infty \}.$$

Define

$$E_x := \{ tX \in T_x M : 0 \leq t < t_c(X), X \in T_x M \text{ with } |X| = 1 \} \subset T_x M.$$

One can show that $\text{Cut}(x) = \exp_x(\partial E_x)$, $M = \exp_x(E_x) \cup \text{Cut}(x)$ and $\exp_x : E_x \rightarrow \exp_x(E_x)$ is a diffeomorphism. We note that $\text{Cut}(x)$ is closed and has measure zero. For any $x \notin \text{Cut}(y)$ with $x \neq y$, then d_y is smooth at x and the Gauss lemma implies that

$$\nabla d_y(x) = - \frac{\exp_x^{-1}(y)}{|\exp_x^{-1}(y)|}$$

and

$$\nabla(d_y^2/2)(x) = -\exp_x^{-1}(y).$$

Let the Riemannian curvature tensor be defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

where ∇ stands for the Levi-Civita connection. For a unit vector $e \in T_x M$, $R(e)$ will denote the Ricci transform of $T_x M$ into itself given by $R(e)X := R(X, e)e$.

For $u \in C^\infty(M)$, the Hessian operator $D^2u(x) : T_x M \rightarrow T_x M$ is defined by

$$D^2u(x) \cdot X = \nabla_X \nabla u(x).$$

Let M and N be Riemannian manifolds of dimension n and $\phi : M \rightarrow N$ be smooth. The Jacobian of ϕ is the absolute value of determinant of the differential $d\phi$, i.e.,

$$\text{Jac } \phi(x) := |\det d\phi(x)| \quad \text{for } x \in M.$$

We quote the following lemma from Lemma 3.2 in [Ca], in which the Jacobian of the map $x \mapsto \exp_x(\nabla v(x))$ is computed explicitly.

Lemma 2.1 (Cabr ). *Let v be a smooth function in an open set Ω of M . Define the map $\phi : \Omega \rightarrow M$ by*

$$\phi(x) := \exp_x \nabla v(x).$$

Let $x \in \Omega$ and suppose that $\nabla v(x) \in E_x$. Set $y := \phi(x)$. Then we have

$$\text{Jac } \phi(x) = \text{Jac } \exp_x(\nabla v(x)) \cdot \left| \det D^2 \left(v + d_y^2/2 \right) (x) \right|,$$

where $\text{Jac } \exp_x(\nabla v(x))$ denotes the Jacobian of \exp_x , a map from $T_x M$ to M , at the point $\nabla v(x) \in T_x M$.

Under the condition (3), we have the estimate for Jacobian of the exponential map and Bishop's volume comparison theorem as follows. We state the known results as a lemma. The proof can be found in [K, p. 286] (see also [L]).

Lemma 2.2. *Suppose that M satisfies (3).*

(i) *For any $x \in M$ and $X \in E_x$,*

$$\text{Jac } \exp_x(X) = |\det d \exp_x(X)| \leq 1.$$

(ii) (Bishop) *For any $x \in M$, $\text{Vol}(B_R(x))/R^n$ is nonincreasing with respect to R , where $B_R(x)$ is a geodesic ball of radius R centered at x . Namely,*

$$\frac{\text{Vol}(B_R(x))}{\text{Vol}(B_r(x))} \leq \frac{R^n}{r^n} \quad \text{if } 0 < r < R.$$

In particular, M satisfies the volume doubling property; i.e., $\text{Vol}(B_{2R}(x)) \leq 2^n \text{Vol}(B_R(x))$.

The following is the area formula, which follows easily from the area formula in Euclidean space and a partition of unity.

Lemma 2.3 (Area formula). *For any smooth function $\phi : M \times \mathbb{R} \rightarrow M \times \mathbb{R}$ and any measurable set $E \subset M \times \mathbb{R}$, we have*

$$\int_E \text{Jac } \phi(x, t) dV(x, t) = \int_{M \times \mathbb{R}} \mathcal{H}^0[E \cap \phi^{-1}(y, s)] dV(y, s),$$

where \mathcal{H}^0 is the counting measure.

Notation. *Let us summarize the notations and definitions that will be used.*

- *Let $r > 0, \rho > 0, z_o \in M$ and $t_o \in \mathbb{R}$. We denote*

$$K_{r,\rho}(z_o, t_o) := B_r(z_o) \times (t_o - \rho, t_o],$$

where $B_r(z_o)$ is a geodesic ball of radius r centered at z_o .

- *We denote $K_r(z_o, t_o) := K_{r,r^2}(z_o, t_o)$.*
- *We say that a constant C is uniform if C depends only on n, λ, Λ and a_L .*
- *We denote $\int_Q f := \frac{1}{\text{Vol}(Q)} \int_Q f$.*
- *We denote $|Q| := \text{Vol}(Q)$.*
- *We denote the trace by tr .*

3. KEY LEMMA

In this section, we obtain Aleksandrov-Bakelman-Pucci-Krylov-Tso type estimate (Lemma 3.2) for parabolic Harnack inequalities. We begin with direct computation of the Jacobian of the parabolic normal map Φ below, which is a parabolic analogue of Lemma 2.1.

Lemma 3.1. *Let v be a smooth function in an open set K of $M \times \mathbb{R}$. Define the map $\phi : K \rightarrow M$ by*

$$\phi(x, t) := \exp_x \nabla_x v(x, t)$$

and the map $\Phi : K \rightarrow M \times \mathbb{R}$ by

$$\Phi(x, t) := \left(\phi(x, t), -\frac{1}{2} d(x, \phi(x, t))^2 - v(x, t) \right).$$

Let $(x, t) \in K$ and assume that $\nabla_x v(x, t) \in E_x$. Set $y := \phi(x, t)$. Then

$$\text{Jac } \Phi(x, t) = \text{Jac } \exp_x(\nabla_x v(x, t)) \cdot \left| (-v_t) \det \left(D_x^2 \left(v + d_y^2/2 \right) \right) \right|,$$

where $\text{Jac } \exp_x(\nabla_x v(x, t))$ denotes the Jacobian of \exp_x at the point $\nabla_x v(x, t) \in T_x M$.

Proof. We may assume that $\nabla_x v(x, t) \neq 0$, which is equivalent to $x \neq y$. Let $(\xi, \sigma) \in T_x M \times \mathbb{R} \setminus \{(0, 0)\}$ and let $\gamma = (\gamma_1, \gamma_2)$ be the geodesic with $\gamma(0) = (x, t)$ and $\gamma'(0) = (\xi, \sigma)$. We note that $\gamma_1(\tau) = \exp_x \tau \xi$ and $\gamma_2(\tau) = t + \sigma \tau$. Set

$$Y(s, \tau) := \exp_{\gamma_1(\tau)} [s \nabla_x v(\gamma(\tau))].$$

Consider the family of geodesics (in the parameter s)

$$\Pi(s, \tau) := \left(Y(s, \tau), \gamma_2(\tau) - s \left\{ \frac{1}{2} d(\gamma_1(\tau), \phi(\gamma(\tau)))^2 + v(\gamma(\tau)) + \gamma_2(\tau) \right\} \right)$$

that joins $\Pi(0, \tau) = \gamma(\tau)$ to $\Pi(1, \tau) = \Phi(\gamma(\tau))$. Then we define

$$J(s) := \frac{\partial}{\partial \tau} \Big|_{\tau=0} \Pi(s, \tau),$$

which is a Jacobi field along

$$X(s) := \left(\exp_x(s \nabla_x v(x, t)), t - s \left\{ \frac{1}{2} d(x, \phi(x, t))^2 + v(x, t) + t \right\} \right).$$

Simple computation says that

$$J(0) = (\xi, \sigma) \quad \text{and} \quad J(1) = \frac{\partial}{\partial \tau} \Big|_{\tau=0} \Phi(\gamma(\tau)) = d\Phi(x, t) \cdot (\xi, \sigma).$$

We also have

$$\begin{aligned} D_s J(0) = & \left(D_x^2 v(x, t) \xi + \sigma \nabla_x v_t(x, t), -\sigma v_t(x, t) - \sigma \right. \\ & \left. - \left\langle \nabla_x \left(d_x^2/2 \right)(y), d \exp_x(\nabla_x v(x, t)) \cdot \left(D_x^2 \left(v + d_y^2/2 \right)(x, t) \cdot \xi + \sigma \nabla_x v_t(x, t) \right) \right\rangle \right). \end{aligned}$$

In fact, we have

$$\begin{aligned}
D_s J(0) &= D_s \frac{\partial \Pi}{\partial \tau} \Big|_{s=0, \tau=0} = D_\tau \frac{\partial \Pi}{\partial s} \Big|_{s=0, \tau=0} \\
&= D_\tau \Big|_{\tau=0} \left(\nabla_x v(\gamma(\tau)), -\frac{1}{2} d(\gamma_1(\tau), \phi(\gamma(\tau)))^2 - v(\gamma(\tau)) - \gamma_2(\tau) \right) \\
&= \left(D_x^2 v(x, t) \cdot \xi + \sigma \nabla_x v_t(x, t), \right. \\
&\quad \left. - \left\langle \nabla_x (d_y^2/2)(x), \xi \right\rangle - \left\langle \nabla_x (d_x^2/2)(\phi(x, t)), \frac{\partial}{\partial \tau} \phi(\gamma(\tau)) \Big|_{\tau=0} \right\rangle - \langle \nabla_x v(x, t), \xi \rangle - \sigma v_t - \sigma \right) \\
&= \left(D_x^2 v \cdot \xi + \sigma \nabla_x v_t, - \left\langle \nabla_x (d_x^2/2)(\phi(x, t)), \frac{\partial}{\partial \tau} \phi(\gamma(\tau)) \Big|_{\tau=0} \right\rangle - \sigma v_t - \sigma \right),
\end{aligned}$$

since $\nabla_x (d_y^2/2)(x) = -\exp_x^{-1}(y) = -\nabla_x v(x, t)$. Then we use Lemma 2.1 to obtain

$$\frac{\partial}{\partial \tau} \phi(\gamma(\tau)) \Big|_{\tau=0} = d \exp_x(\nabla_x v(x, t)) \cdot \left(D_x^2 \left(v + d_y^2/2 \right)(x, t) \cdot \xi + \sigma \nabla_x v_t(x, t) \right).$$

On the other hand, consider the Jacobi field $J_{\xi, \sigma}$ along $X(s)$ satisfying

$$J_{\xi, \sigma}(0) = (\xi, \sigma) \quad \text{and} \quad J_{\xi, \sigma}(1) = (0, 0).$$

Then we can check that

$$J_{\xi, \sigma}(s) = \frac{\partial}{\partial \tau} \Psi \Big|_{\tau=0} \quad \text{and} \quad D_s J_{\xi, \sigma}(0) = \left(-D_x^2 \left(d_y^2/2 \right)(x) \cdot \xi, -\sigma \right),$$

where

$$\Psi(s, \tau) := \left(\exp_{\gamma_1(\tau)} s \exp_{\gamma_1(\tau)}^{-1} \phi(x, t), \gamma_2(\tau) - s \left\{ \frac{1}{2} d(x, \phi(x, t))^2 + v(x, t) + \gamma_2(\tau) \right\} \right).$$

(We refer [Ca, Lemma 3.2] for the proof.)

Define $\tilde{J}_{\xi, \sigma} := J - J_{\xi, \sigma}$. The Jacobi field $\tilde{J}_{\xi, \sigma}$ along $X(s)$ satisfying

$$\tilde{J}_{\xi, \sigma}(0) = (0, 0) \quad \text{and} \quad D_s \tilde{J}_{\xi, \sigma}(0) = D_s J(0) - D_s J_{\xi, \sigma}(0)$$

is written by

$$d \exp_{(x, t)}(s X'(0)) \cdot (s D_s \tilde{J}_{\xi, \sigma}(0)).$$

Therefore, we have

$$J(1) = \tilde{J}_{\xi, \sigma}(1) = d \exp_{(x, t)} \left(\nabla_x v(x, t), -\frac{1}{2} d(x, y)^2 - v(x, t) - t \right) \cdot (D_s J(0) - D_s J_{\xi, \sigma}(0)),$$

which means

$$\begin{aligned}
d\Phi(x, t) \cdot (\xi, \sigma) &= d \exp_{(x, t)} \left(\nabla_x v(x, t), -\frac{1}{2} d(x, y)^2 - v(x, t) - t \right) \cdot \\
&\quad \left(D_x^2 \left(v + d_y^2/2 \right)(x, t) \cdot \xi + \sigma \nabla_x v_t(x, t), \right. \\
&\quad \left. -\sigma v_t - \left\langle \nabla_x \left(d_x^2/2 \right)(y), d \exp_x(\nabla_x v(x, t)) \cdot \left(D_x^2 \left(v + d_y^2/2 \right)(x, t) \cdot \xi + \sigma \nabla_x v_t(x, t) \right) \right\rangle \right) \\
&= \left(d \exp_x(\nabla_x v(x, t)) \cdot \left(D_x^2 \left(v + d_y^2/2 \right)(x, t) \cdot \xi + \sigma \nabla_x v_t(x, t) \right), \right. \\
&\quad \left. -\sigma v_t - \left\langle \nabla_x \left(d_x^2/2 \right)(y), d \exp_x(\nabla_x v(x, t)) \cdot \left(D_x^2 \left(v + d_y^2/2 \right)(x, t) \cdot \xi + \sigma \nabla_x v_t(x, t) \right) \right\rangle \right).
\end{aligned}$$

To calculate the Jacobian of Φ , we introduce an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_x M$ and an orthonormal basis $\{\bar{e}_1, \dots, \bar{e}_n\}$ of $T_y M = T_{\exp_x \nabla v(x,t)} M$. By setting for $i, j = 1, \dots, n$,

$$\begin{aligned} A_{ij} &:= \left\langle \bar{e}_i, d \exp_x (\nabla_x v(x, t)) \cdot \left(D_x^2 \left(v + d_y^2/2 \right) (x, t) e_j \right) \right\rangle, \\ b_i &:= \left\langle \bar{e}_i, d \exp_x (\nabla_x v(x, t)) \cdot \nabla v_t(x, t) \right\rangle, \\ c_i &:= \left\langle \bar{e}_i, \nabla_x (d_x^2/2)(y) \right\rangle, \end{aligned}$$

the Jacobian matrix of Φ at (x, t) is

$$\begin{pmatrix} A_{ij} & b_i \\ -c_k A_{kj} & -v_t - b_k c_k \end{pmatrix}.$$

Lastly, we use the row operations to deduce that

$$\text{Jac } \Phi(x, t) = \left| \det \begin{pmatrix} A_{ij} & b_i \\ 0 & -v_t \end{pmatrix} \right| = |(-v_t) \det(A_{ij})|.$$

This completes the proof. \square

The following lemma will play a key role to estimate sublevel sets of u in Lemma 4.3 and then to prove a decay estimate of the distribution function of u in Lemma 6.1. This ABP-type lemma corresponds to [Ca, Lemma 4.1].

Lemma 3.2. *Suppose that M satisfies the condition (4). Let $z_o \in M$, $R > 0$, and $0 < \eta < 1$. Let u be a smooth function in $K_{\alpha_1 R, \alpha_2 R^2}(z_o, 0) \subset M \times \mathbb{R}$ satisfying*

$$(7) \quad u \geq 0 \quad \text{in} \quad K_{\alpha_1 R, \alpha_2 R^2}(z_o, 0) \setminus K_{\beta_1 R, \beta_2 R^2}(z_o, 0) \quad \text{and} \quad \inf_{K_{2R}(z_o, 0)} u \leq 1,$$

where $\alpha_1 := \frac{11}{\eta}$, $\alpha_2 := 4 + \eta^2 + \frac{\eta^4}{4}$, $\beta_1 := \frac{9}{\eta}$, and $\beta_2 := 4 + \eta^2$. Then we have

$$(8) \quad |B_R(z_o)| \cdot R^2 \leq C(\eta, n, \lambda) \int_{\{u \leq M_\eta\} \cap K_{\beta_1 R, \beta_2 R^2}(z_o, 0)} \left\{ \left(R^2 \mathcal{L}u + a_L + \Lambda + 1 \right)^+ \right\}^{n+1}$$

where the constant $M_\eta > 0$ depends only on $\eta > 0$ and $C(\eta, n, \lambda) > 0$ depends only on η, n and λ .

Proof. For any $\bar{y} \in B_R(z_o)$, we define

$$w_{\bar{y}}(x, t) := \frac{1}{2} R^2 u(x, t) + \frac{1}{2} d_{\bar{y}}^2(x) - C_\eta t, \quad C_\eta := \frac{6}{\eta^2}.$$

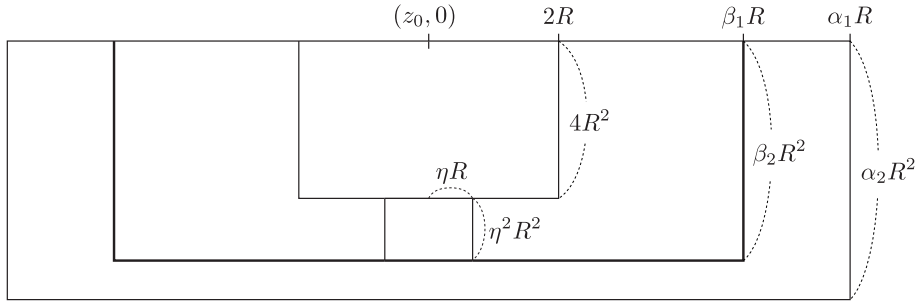


FIGURE 1. $\alpha_1 := \frac{11}{\eta}$, $\alpha_2 := 4 + \eta^2 + \frac{\eta^4}{4}$, $\beta_1 := \frac{9}{\eta}$, $\beta_2 := 4 + \eta^2$.

From the assumption (7), it is easy to check that

$$\inf_{K_{2R}(z_o, 0)} w_{\bar{y}} \leq \left(5 + \frac{24}{\eta^2}\right) R^2 =: A_\eta R^2,$$

and

$$w_{\bar{y}} \geq \left(6 + \frac{24}{\eta^2}\right) R^2 = (A_\eta + 1)R^2 \quad \text{on } K_{\alpha_1 R, \alpha_2 R^2}(z_o, 0) \setminus K_{\beta_1 R, \beta_2 R^2}(z_o, 0).$$

From the above observation, for any $(\bar{y}, \bar{h}) \in B_R(z_o) \times (A_\eta R^2, (A_\eta + 1)R^2)$, we can find a time $\bar{t} \in (-\beta_2 R^2, 0)$ such that

$$\bar{h} = \inf_{B_{\beta_1 R}(z_o) \times (-\beta_2 R^2, \bar{t}]} w_{\bar{y}}(z, \tau) = w_{\bar{y}}(\bar{x}, \bar{t}),$$

where the infimum is achieved at an interior point \bar{x} of $B_{\beta_1 R}(z_o)$. By the same argument as in [Ca, pp. 637-638], we have the following relation:

$$\bar{y} = \exp_x \nabla_x \left(\frac{1}{2} R^2 u \right) (\bar{x}, \bar{t}).$$

Now, we consider the map $\Phi : K_{\alpha_1 R, \alpha_2 R^2}(z_o, 0) \rightarrow M \times \mathbb{R}$ (with $v(x, t) = \frac{1}{2} R^2 u(x, t) - C_\eta t$ in Lemma 3.1) defined as

$$\Phi(x, t) := \left(\exp_x \nabla_x \left(\frac{1}{2} R^2 u \right) (x, t), -\frac{1}{2} d \left(x, \exp_x \nabla_x \left(\frac{1}{2} R^2 u \right) (x, t) \right)^2 - \frac{1}{2} R^2 u(x, t) + C_\eta t \right).$$

Define a set

$$E := \left\{ (x, t) \in K_{\beta_1 R, \beta_2 R^2}(z_o, 0) : \exists y \in B_R(z_o) \text{ s.t. } w_y(x, t) = \inf_{B_{\beta_1 R}(z_o) \times (-\beta_2 R^2, t]} w_y \leq (A_\eta + 1)R^2 \right\}.$$

The set E is a subset of the contact set in $K_{\beta_1 R, \beta_2 R^2}(z_o, 0)$ that contains a point (x, t) , where a concave paraboloid $-\frac{1}{2} d_y^2(x) + C_\eta t + C$ (for some C) touches $\frac{1}{2} R^2 u$ from below. Thus we have proved that for any $(y, s) \in B_R(z_o) \times (-(A_\eta + 1)R^2, -A_\eta R^2)$, there is at least one $(x, t) \in E$ such that $(y, s) = \Phi(x, t)$, namely,

$$B_R(z_o) \times (-(A_\eta + 1)R^2, -A_\eta R^2) \subset \Phi(E).$$

So Area formula gives

$$(9) \quad |B_R(z_o)| \cdot R^2 \leq \int_{M \times \mathbb{R}} \mathcal{H}^0 [E \cap \Phi^{-1}(y, s)] dV(y, s) = \int_E \text{Jac } \Phi(x, t) dV(x, t).$$

We notice that for $(x, t) \in E$ and $y \in B_R(z_o)$, $w_y(x, t) = \frac{1}{2} R^2 u(x, t) + \frac{1}{2} d_y^2(x) - C_\eta t \leq (A_\eta + 1)R^2$ and hence $u(x, t) \leq 2(A_\eta + 1) =: M_\eta$ for $(x, t) \in E$.

Lastly, we claim that for $(x, t) \in E$,

$$(10) \quad \text{Jac } \Phi(x, t) \leq \frac{1}{(n+1)^{n+1} \lambda^n} \left\{ \left(\frac{1}{2} R^2 \mathcal{L} u(x, t) + a_L + \Lambda + C_\eta \right)^+ \right\}^{n+1}.$$

Fix $(x, t) \in E$ and $y \in B_R(z_o)$ to satisfy

$$w_y(x, t) = \inf_{B_{\beta_1 R}(z_o) \times (-\beta_2 R^2, t]} w_y.$$

We recall that $y = \exp_x \nabla_x \left(\frac{1}{2} R^2 u \right) (x, t)$ (see [Ca, pp. 637-638]).

If x is not a cut point of y , then Lemma 3.1 (with $v(x, t) = \frac{1}{2}R^2u(x, t) - C_\eta t$) and Lemma 2.2 (i) imply that

$$\text{Jac } \Phi(x, t) \leq \left| \left(-\frac{1}{2}R^2u_t + C_\eta \right) \det \left(D_x^2 \left(\frac{1}{2}R^2u + \frac{1}{2}d_y^2 \right) \right) (x, t) \right|.$$

Since the minimum of w_y in $B_{\beta_1 R}(z_o) \times (-\beta_2 R^2, t]$ is achieved at (x, t) , we have

$$0 \leq D_x^2 w_y(x, t) = D_x^2 \left(\frac{1}{2}R^2u + \frac{1}{2}d_y^2 \right) \quad \text{and} \quad 0 \geq \partial_t w_y(x, t) = \frac{1}{2}R^2u_t - C_\eta,$$

where $D_x^2 w_y(x, t) \geq 0$ means that the Hessian of w_y at (x, t) is positive semidefinite. Therefore, by using the geometric and arithmetic means inequality, we get

$$\begin{aligned} \text{Jac } \Phi(x, t) &\leq \left(-\frac{1}{2}R^2u_t + C_\eta \right) \det \left(D_x^2 \left(\frac{1}{2}R^2u + \frac{1}{2}d_y^2 \right) \right) (x, t) \\ &\leq \frac{1}{\lambda^n} \left(-\frac{1}{2}R^2u_t + C_\eta \right) \det A_{x,t} \det \left(D_x^2 \left(\frac{1}{2}R^2u + \frac{1}{2}d_y^2 \right) \right) \\ &\leq \frac{1}{(n+1)^{n+1}\lambda^n} \left\{ \text{tr} \left(A_{x,t} \circ D_x^2 \left(\frac{1}{2}R^2u + \frac{1}{2}d_y^2 \right) \right) - \frac{1}{2}R^2u_t + C_\eta \right\}^{n+1} \\ &= \frac{1}{(n+1)^{n+1}\lambda^n} \left\{ \frac{1}{2}R^2 \mathcal{L}u(x, t) + L \left[\frac{1}{2}d_y^2 \right] + C_\eta \right\}^{n+1} \\ &\leq \frac{1}{(n+1)^{n+1}\lambda^n} \left\{ \frac{1}{2}R^2 \mathcal{L}u + a_L + \Lambda + C_\eta \right\}^{n+1} \\ &= \frac{1}{(n+1)^{n+1}\lambda^n} \left\{ \left(\frac{1}{2}R^2 \mathcal{L}u + a_L + \Lambda + C_\eta \right)^+ \right\}^{n+1}, \end{aligned}$$

where we used

$$L \left[d_y^2/2 \right] = d_y L d_y + \langle A_{x,t} \nabla d_y, \nabla d_y \rangle \leq a_L + \Lambda |\nabla d_y|^2.$$

When x is a cut point of y , we make use of upper barrier technique due to Calabi [Cal]. Since $y = \exp_x \nabla_x \left(\frac{1}{2}R^2u \right) (x, t)$, x is not a cut point of $y_\sigma := \phi_\sigma(x, t) := \exp_x \nabla_x \left(\frac{\sigma}{2}R^2u \right) (x, t)$ for $0 \leq \sigma < 1$. Now we consider

$$\Phi_\sigma(z, \tau) := \left(\phi_\sigma(z, \tau), -\frac{\sigma}{2}R^2u(z, \tau) - \frac{1}{2}d(z, \phi_\sigma(z, \tau))^2 + C_\eta \tau \right)$$

instead of Φ since $\text{Jac } \Phi(x, t) = \lim_{\sigma \uparrow 1} \text{Jac } \Phi_\sigma(x, t)$. As before, we have

$$\text{Jac } \Phi_\sigma(x, t) \leq \left| \left(-\frac{\sigma}{2}R^2u_t + C_\eta \right) \det \left(D_x^2 \left(\frac{\sigma}{2}R^2u + \frac{1}{2}d_{y_\sigma}^2 \right) \right) (x, t) \right|.$$

We note that

$$\begin{aligned} &\liminf_{\sigma \uparrow 1} \left| \left(-\frac{\sigma}{2}R^2u_t + C_\eta \right) \det \left(D_x^2 \left(\frac{\sigma}{2}R^2u + \frac{1}{2}d_{y_\sigma}^2 \right) \right) (x, t) \right| \\ &= \liminf_{\sigma \uparrow 1} \left| \left(-\partial_t w_{y_\sigma}(x, t) \right) \det \left(D_x^2 w_{y_\sigma} \right) (x, t) \right| \end{aligned}$$

for $w_{y_\sigma}(z, \tau) := \frac{1}{2}R^2u(z, \tau) + \frac{1}{2}d_{y_\sigma}^2(z) - C_\eta\tau$. According to the triangle inequality, we have

$$\begin{aligned} w_y(z, \tau) &\leq \frac{1}{2}R^2u(z, \tau) + \frac{1}{2}\left(d_{y_\sigma}(z) + d(y_\sigma, y)\right)^2 - C_\eta\tau \\ &= w_{y_\sigma}(z, \tau) + d(y_\sigma, y)d_{y_\sigma}(z) + \frac{1}{2}d(y_\sigma, y)^2, \end{aligned}$$

where the equality holds at $(z, \tau) = (x, t)$. Since w_y has the minimum at (x, t) in $B_{\beta_1 R}(z_o) \times (-\beta_2 R^2, t]$, the minimum of $w_{y_\sigma}(z, \tau) + d(y_\sigma, y)d_{y_\sigma}(z)$ (in $B_{\beta_1 R}(z_o) \times (-\beta_2 R^2, t]$) is also achieved at (x, t) , that implies that

$$D_x^2\left(w_{y_\sigma} + d(y_\sigma, y)d_{y_\sigma}\right)(x, t) \geq 0, \quad \partial_t w_{y_\sigma}(x, t) \leq 0.$$

To bound $D^2 y_\sigma(x)$ uniformly in $\sigma \in [1/2, 1)$, we recall the Hessian comparison theorem (see [S],[SY]): Let $-k^2$ ($k > 0$) be a lower bound of sectional curvature along the minimal geodesic joining x and y . Then for $0 < \sigma < 1$,

$$D^2 d_{y_\sigma}(x) \leq k \coth(kd_{y_\sigma}(x)) \text{Id}$$

and hence we find a constant $N > 0$ independent of σ such that

$$D^2 d_{y_\sigma}(x) \leq N \text{Id} \quad \text{for } \frac{1}{2} \leq \sigma < 1.$$

Following the above argument, for $\frac{1}{2} \leq \sigma < 1$, we obtain

$$\begin{aligned} 0 &\leq \liminf_{\sigma \uparrow 1} \left(-\partial_t w_{y_\sigma}(x, t)\right) \det\left(D_x^2 w_{y_\sigma} + d(y_\sigma, y)D^2 d_{y_\sigma}\right)(x, t) \\ &\leq \liminf_{\sigma \uparrow 1} \left(-\partial_t w_{y_\sigma}(x, t)\right) \det\left(D_x^2 w_{y_\sigma} + d(y_\sigma, y)N \text{Id}\right)(x, t) \\ &\leq \liminf_{\sigma \uparrow 1} \frac{1}{(n+1)^{n+1}\lambda^n} \left\{ \frac{1}{2}R^2 \mathcal{L}u + a_L + \Lambda + C_\eta + d(y_\sigma, y)n\Lambda N \right\}^{n+1} \\ &\leq \frac{1}{(n+1)^{n+1}\lambda^n} \left\{ \left(\frac{1}{2}R^2 \mathcal{L}u + a_L + \Lambda + C_\eta \right)^+ \right\}^{n+1}. \end{aligned}$$

Then we deduce that

$$\text{Jac } \Phi(x, t) \leq \frac{1}{(n+1)^{n+1}\lambda^n} \left\{ \left(\frac{1}{2}R^2 \mathcal{L}u(x, t) + a_L + \Lambda + C_\eta \right)^+ \right\}^{n+1}$$

since

$$\begin{aligned} \liminf_{\sigma \uparrow 1} \left| \det\left(D_x^2 w_{y_\sigma}\right)(x, t) \right| &= \liminf_{\sigma \uparrow 1} \left| \det\left(D_x^2 w_{y_\sigma} + d(y_\sigma, y)N \text{Id}\right)(x, t) \right| \\ &= \liminf_{\sigma \uparrow 1} \det\left(D_x^2 w_{y_\sigma} + d(y_\sigma, y)N \text{Id}\right)(x, t). \end{aligned}$$

We conclude that (10) is true for $(x, t) \in E$. Therefore the estimate (8) follows from (9) since $E \subset \{u \leq M_\eta\} \cap K_{\beta_1 R, \beta_2 R^2}(z_o, 0)$. \square

4. BARRIER FUNCTIONS

We modify the barrier function of [W] to construct a barrier function in the Riemannian case. First, we fix some constants that will be used frequently (see Figure 1); for a given $0 < \eta < 1$,

$$\alpha_1 := \frac{11}{\eta}, \quad \alpha_2 := 4 + \eta^2 + \frac{\eta^4}{4}, \quad \beta_1 := \frac{9}{\eta} \quad \text{and} \quad \beta_2 := 4 + \eta^2.$$

Lemma 4.1. *Suppose that M satisfies the condition (4). Let $z_o \in M$, $R > 0$ and $0 < \eta < 1$. There exists a continuous function $v_\eta(x, t)$ in $K_{\alpha_1 R, \alpha_2 R^2}(z_o, \beta_2 R^2)$, which is smooth in $(M \setminus \text{Cut}(z_o)) \cap K_{\alpha_1 R, \alpha_2 R^2}(z_o, \beta_2 R^2)$ such that*

- (i) $v_\eta(x, t) \geq 0$ in $K_{\alpha_1 R, \alpha_2 R^2}(z_o, \beta_2 R^2) \setminus K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2)$,
- (ii) $v_\eta(x, t) \leq 0$ in $K_{2R}(z_o, \beta_2 R^2)$,
- (iii) $R^2 \mathcal{L} v_\eta + a_L + \Lambda + 1 \leq 0$ a.e. in $K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2) \setminus K_{\frac{\eta}{2} R}(z_o, \frac{\eta^2}{4} R^2)$,
- (iv) $R^2 \mathcal{L} v_\eta \leq C_\eta$ a.e. in $K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2)$,
- (v) $v_\eta(x, t) \geq -C_\eta$ in $K_{\alpha_1 R, \alpha_2 R^2}(z_o, \beta_2 R^2)$.

Here, the constant $C_\eta > 0$ depends only on $\eta, n, \lambda, \Lambda, a_L$ (independent of R and z_o).

Proof. Fix $0 < \eta < 1$. Consider

$$h(s, t) := -Ae^{-mt} \left(1 - \frac{s}{\beta_1^2}\right)^l \frac{1}{(4\pi t)^{n/2}} \exp\left(-\alpha \frac{s}{t}\right) \quad \text{for } t > 0,$$

as in Lemma 3.22 of [W] and define

$$\psi(s, t) := h(s, t) + (a_L + \Lambda + 1)t \quad \text{in } [0, \beta_1^2] \times [0, \beta_2] \setminus [0, \frac{\eta^2}{4}] \times [0, \frac{\eta^2}{4}],$$

where the positive constants A, m, l, α (depending only on $\eta, n, \lambda, \Lambda, a_L$) will be chosen later. In particular, l will be an odd number in \mathbb{N} . We extend ψ smoothly in $[0, \alpha_1^2] \times [-\frac{\eta^4}{4}, \beta_2]$ to satisfy

$$\begin{aligned} \psi &\geq 0 \quad \text{on } [0, \alpha_1^2] \times [-\frac{\eta^4}{4}, \beta_2] \setminus [0, \beta_1^2] \times [0, \beta_2], \\ \psi &\geq -C_\eta \quad \text{on } [0, \alpha_1^2] \times [-\frac{\eta^4}{4}, \beta_2], \end{aligned}$$

and

$$\sup_{[0, \beta_1^2] \times [0, \beta_2]} \{2a_L |\partial_s \psi| + \Lambda (2|\partial_s \psi| + 4s|\partial_{ss} \psi|) + |\partial_t \psi|\} (s, t) < C_\eta$$

for some $C_\eta > 0$. We also assume that $\psi(s, t)$ is nondecreasing with respect to s in $[0, \alpha_1^2] \times [-\frac{\eta^4}{4}, \beta_2]$. We define

$$v_\eta(x, t) = v(x, t) := \psi\left(\frac{d_{z_o}^2(x)}{R^2}, \frac{t}{R^2}\right) \quad \text{for } (x, t) \in K_{\alpha_1 R, \alpha_2 R^2}(z_o, \beta_2 R^2),$$

where d_{z_o} is the distance function to z_o . Properties (i) and (v) are trivial.

We denote $d_{z_o}(x)$ and $h\left(\frac{d_{z_o}^2(x)}{R^2}, \frac{t}{R^2}\right)$ by $d(x)$ and $\phi(x, t)$ for simplicity and we notice that for $(x, t) \in K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2) \setminus K_{\frac{\eta}{2} R}(z_o, \frac{\eta^2}{4} R^2)$,

$$v(x, t) = h\left(\frac{d^2(x)}{R^2}, \frac{t}{R^2}\right) + (a_L + \Lambda + 1)\frac{t}{R^2} = \phi(x, t) + (a_L + \Lambda + 1)\frac{t}{R^2}$$

and $\phi(x, t)$ is negative in $K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2)$.

Now, we claim that

$$(11) \quad \mathcal{L} \phi \leq 0 \quad \text{a.e. in } K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2) \setminus K_{\frac{\eta}{2} R}(z_o, \frac{\eta^2}{4} R^2).$$

Once (11) is proved, then property (iii) follows from the simple calculation that $R^2 \mathcal{L} \left[(a_L + \Lambda + 1) \frac{t}{R^2} \right] = -(a_L + \Lambda + 1)$ in $K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2)$. Now we use the identity

$$\mathcal{L}[\varphi(u(x), t)] = \partial_u \varphi(u, t) \mathcal{L} u + \partial_{uu} \varphi(u, t) \langle A_{x,t} \nabla u, \nabla u \rangle - \partial_t \varphi(u, t)$$

to obtain

$$\begin{aligned}\mathcal{L}\phi &= \frac{2d}{R^2} \partial_s h \left(\frac{d^2}{R^2}, \frac{t}{R^2} \right) Ld \\ &\quad + \left\{ \frac{2}{R^2} \partial_s h + \frac{4d^2}{R^4} \partial_{ss} h \right\} \left(\frac{d^2}{R^2}, \frac{t}{R^2} \right) \langle A_{x,t} \nabla d, \nabla d \rangle - \frac{1}{R^2} \partial_t h \left(\frac{d^2}{R^2}, \frac{t}{R^2} \right).\end{aligned}$$

Since $d \cdot Ld \leq a_L$ and $\lambda \leq \langle A_{x,t} \nabla d, \nabla d \rangle \leq \Lambda$ in $M \setminus \text{Cut}(z_o)$, we have that

$$\begin{aligned}& \frac{(\beta_1^2 R^2 - d^2)^2}{(-\phi)} \mathcal{L}\phi \\ &= (\beta_1^2 R^2 - d^2) \left\{ 2l + (\beta_1^2 R^2 - d^2) \frac{2\alpha}{t} \right\} d \mathcal{L}d \\ &\quad - \left\{ l(l-1)4d^2 + 2l(\beta_1^2 R^2 - d^2) \frac{4\alpha d^2}{t} + (\beta_1^2 R^2 - d^2)^2 \frac{4\alpha^2 d^2}{t^2} \right\} \langle A_{x,t} \nabla d, \nabla d \rangle \\ &\quad + (\beta_1^2 R^2 - d^2) \left\{ 2l + (\beta_1^2 R^2 - d^2) \frac{2\alpha}{t} \right\} \langle A_{x,t} \nabla d, \nabla d \rangle \\ &\quad + (\beta_1^2 R^2 - d^2)^2 \frac{\alpha d^2}{t^2} - (\beta_1^2 R^2 - d^2)^2 \left(\frac{n}{2t} + \frac{m}{R^2} \right) \\ &\leq 2l(\beta_1^2 R^2 - d^2)(a_L + \Lambda) + (\beta_1^2 R^2 - d^2)^2 \left\{ \frac{2\alpha}{t}(a_L + \Lambda) + \frac{\alpha d^2}{t^2} \right\} \\ &\quad - l(l-1)4d^2 \lambda - (\beta_1^2 R^2 - d^2)^2 \left(\frac{4\alpha^2 d^2}{t^2} \lambda + \frac{n}{2t} \right) \\ &\quad - 2l(\beta_1^2 R^2 - d^2) \frac{4\alpha d^2}{t} \lambda - \frac{m}{R^2} (\beta_1^2 R^2 - d^2)^2 \quad \text{a.e. in } K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2).\end{aligned}$$

By choosing

$$(12) \quad \begin{aligned}\alpha &:= \frac{1}{4\lambda}, \quad \frac{2\beta_1^2}{\eta^2 \lambda} (a_L + \Lambda) + 1 \leq l := 2l' + 1 \quad (\text{for some } l' \in \mathbb{N}), \\ m &:= 2 \cdot \max \left\{ \frac{8\alpha}{\eta^2} (a_L + \Lambda), \frac{2l(a_L + \Lambda)}{\beta_1^2 - \frac{\eta^2}{4}} \right\},\end{aligned}$$

we deduce

$$\frac{(\beta_1^2 R^2 - d^2)^2}{(-\phi)} \mathcal{L}\phi \leq 0 \quad \text{a.e. in } K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2) \setminus K_{\frac{\eta}{2} R}(z_o, \frac{\eta^2}{4} R^2).$$

Indeed, we divide the domain $K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2) \setminus K_{\frac{\eta}{2} R}(z_o, \frac{\eta^2}{4} R^2)$ into three regions such that

$$K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2) \setminus K_{\frac{\eta}{2} R}(z_o, \frac{\eta^2}{4} R^2) =: A_1 \cup A_2 \cup A_3,$$

where $A_1 := \{0 \leq \frac{t}{R^2} \leq \frac{\eta^2}{4}, \frac{\eta}{2} \leq \frac{d}{R} \leq \beta_1\}$, $A_2 := \{\frac{\eta^2}{4} \leq \frac{t}{R^2} \leq \beta_2, \frac{\eta}{2} \leq \frac{d}{R} \leq \beta_1\}$ and $A_3 := \{\frac{\eta^2}{4} \leq \frac{t}{R^2} \leq \beta_2, 0 \leq \frac{d}{R} \leq \frac{\eta}{2}\}$. We can check that

$$\frac{(\beta_1^2 R^2 - d^2)^2}{(-\phi)} \mathcal{L}\phi \leq 0 \quad \text{a.e. in } K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2) \setminus K_{\frac{\eta}{2} R}(z_o, \frac{\eta^2}{4} R^2)$$

by choosing α and l large in A_1 , m large in A_2 and A_3 as in (12). Therefore, we have proved (11).

From the assumption on ψ , we have that for a.e. $(x, t) \in K_{\beta_1 R, \beta_2 R^2}(z_o, \beta_2 R^2)$,

$$\begin{aligned} R^2 \mathcal{L}v(x, t) &= 2\partial_s \psi \left(\frac{d^2}{R^2}, \frac{t}{R^2} \right) d \cdot Ld + \left\{ 2\partial_s \psi + \frac{4d^2}{R^2} \partial_{ss} \psi \right\} \left(\frac{d^2}{R^2}, \frac{t}{R^2} \right) \langle A_{x,t} \nabla d, \nabla d \rangle \\ &\quad - \partial_t \psi \left(\frac{d^2}{R^2}, \frac{t}{R^2} \right) \\ &\leq \sup_{[0, \beta_1^2] \times [0, \beta_2]} \{ 2a_L \partial_s \psi + \Lambda (2\partial_s \psi + 4s |\partial_{ss} \psi|) + |\partial_t \psi| \} (s, t) < C_\eta. \end{aligned}$$

This proves property (iv).

In order to show (ii), we take $A > 0$ large enough so that for $(x, t) \in K_{2R}(z_o, \beta_2 R^2)$,

$$v(x, t) \leq -Ae^{-\beta_2 m} \left(1 - \frac{4}{\beta_1^2} \right)^l \frac{1}{(4\pi\beta_2)^{n/2}} e^{-4\alpha/\eta^2} + (a_L + \Lambda + 1)\beta_2 \leq 0.$$

This finishes the proof of the lemma. \square

Now we apply Lemma 3.2 to $u + v_\eta$ with v_η constructed in Lemma 4.1 and translated in time. Since the barrier function $v_\eta(x, t) = \psi_\eta \left(\frac{d_{z_o}^2(x)}{R^2}, \frac{t}{R^2} \right)$ is not smooth on $\text{Cut}(z_o)$, we need to approximate v_η by a sequence of smooth functions as Cabré's approach at [Ca]. We recall that the cut locus of z_o is closed and has measure zero. It is not hard to verify the following lemma and we just refer to [Ca] Lemmas 5.3, 5.4.

Lemma 4.2. *Let $z_o \in M$, $R > 0$ and let $\psi : \mathbb{R}^+ \times [0, T] \rightarrow \mathbb{R}$ be a smooth function such that $\psi(s, t)$ is nondecreasing with respect to s for any $t \in [0, T]$. Let $v(x, t) := \psi \left(d_{z_o}^2(x), t \right)$. Then there exist a smooth function $0 \leq \zeta(x) \leq 1$ on M satisfying*

$$\zeta \equiv 1 \text{ in } B_{\beta_1 R}(z_o) \text{ and } \text{supp } \zeta \subset B_{\frac{10}{\eta} R}(z_o)$$

and a sequence $\{w_k\}_{k=1}^\infty$ of smooth functions in $M \times [0, T]$ such that

$$\begin{cases} w_k \rightarrow \zeta v & \text{uniformly in } M \times [0, T], \\ \partial_t w_k \rightarrow \zeta \partial_t v & \text{uniformly in } M \times [0, T], \\ D_x^2 w_k \leq C \text{Id} & \text{in } M \times [0, T], \\ D_x^2 w_k \rightarrow D_x^2 v & \text{a.e. in } B_{\beta_1 R}(z_o) \times [0, T], \end{cases}$$

where the constant $C > 0$ is independent of k .

Lemma 4.3. *Suppose that M satisfies the conditions (3),(4). Let $z_o \in M, R > 0$, and $0 < \eta < 1$. Let u be a smooth function such that $\mathcal{L}u \leq f$ in $K_{\alpha_1 R, \alpha_2 R^2}(z_o, 4R^2)$ such that*

$$u \geq 0 \text{ in } K_{\alpha_1 R, \alpha_2 R^2}(z_o, 4R^2) \setminus K_{\beta_1 R, \beta_2 R^2}(z_o, 4R^2)$$

and

$$\inf_{K_{2R}(z_o, 4R^2)} u \leq 1.$$

Then, there exist uniform constants $M_\eta > 1, 0 < \mu_\eta < 1$, and $0 < \varepsilon_\eta < 1$ such that

$$(13) \quad \frac{\left| \{u \leq M_\eta\} \cap K_{\eta R}(z_o, 0) \right|}{\left| K_{\alpha_1 R, \alpha_2 R^2}(z_o, 4R^2) \right|} \geq \mu_\eta,$$

provided

$$(14) \quad R^2 \left(\int_{K_{\alpha_1 R, \alpha_2 R^2}(z_o, 4R^2)} |f^+|^{n+1} \right)^{\frac{1}{n+1}} \leq \varepsilon_\eta,$$

where $M_\eta > 0, 0 < \varepsilon_\eta, \mu_\eta < 1$ depend only on $\eta, n, \lambda, \Lambda$ and a_L .

Proof. Let v_η be the barrier function in Lemma 4.1 after translation in time (by $-\eta^2 R^2$) and let $\{w_k\}_{k=1}^\infty$ be a sequence of smooth functions approximating v_η as in Lemma 4.2. We notice that $u + v_\eta \geq 0$ in $K_{\alpha_1 R, \alpha_2 R^2}(z_o, 4R^2) \setminus K_{\beta_1 R, \beta_2 R^2}(z_o, 4R^2)$ and $\inf_{K_{2R}(z_o, 4R^2)} (u + v_\eta) \leq 1$. Thanks to the uniform convergence of w_k to ζv_η , we consider a sequence $\{\varepsilon_k\}_{k=1}^\infty$ converging to 0 such that $\sup_{K_{2R}(z_o, 4R^2)} w_k \leq \varepsilon_k$ and

$$w_k \geq -\varepsilon_k \text{ in } K_{\alpha_1 R, \alpha_2 R^2}(z_o, 4R^2) \setminus K_{\beta_1 R, \beta_2 R^2}(z_o, 4R^2),$$

and define

$$\bar{w}_k := \frac{u + w_k + \varepsilon_k}{1 + 2\varepsilon_k}.$$

Then \bar{w}_k satisfies the hypotheses of Lemma 3.2 (after translation in time by $4R^2$). Now we replace u by \bar{w}_k in (8) and then the uniform convergence implies that for a given $0 < \delta < 1$, we have

$$|B_R(z_o)|R^2 \leq C(\eta, n, \lambda) \int_{\{u+v_\eta \leq M_\eta + \delta\} \cap K_{\beta_1 R, \beta_2 R^2}(z_o, 4R^2)} \left\{ \left(R^2 \mathcal{L} \bar{w}_k + a_L + \Lambda + 1 \right)^+ \right\}^{n+1}$$

if k is sufficiently large. Since $D_x^2 w_k \leq C \text{Id}$ and $|\partial_t w_k| < C$ uniformly in k on $K_{\beta_1 R, \beta_2 R^2}(z_o, 4R^2)$, we use the dominated convergence theorem to let k go to $+\infty$. Letting δ go to 0, we obtain

$$\begin{aligned} |B_R(z_o)| \cdot R^2 &\leq C(\eta, n, \lambda) \int_{\{u+v_\eta \leq M_\eta\} \cap K_{\beta_1 R, \beta_2 R^2}(z_o, 4R^2)} \left\{ \left(R^2 \mathcal{L}[u + v_\eta] + a_L + \Lambda + 1 \right)^+ \right\}^{n+1} \\ &= C(\eta, n, \lambda) \int_{E_1 \cup E_2} \left\{ \left(R^2 \mathcal{L}[u + v_\eta] + a_L + \Lambda + 1 \right)^+ \right\}^{n+1}, \end{aligned}$$

where $E_1 := \{u + v_\eta \leq M_\eta\} \cap (K_{\beta_1 R, \beta_2 R^2}(z_o, 4R^2) \setminus K_{\eta R}(z_o, 0))$ and $E_2 := \{u + v_\eta \leq M_\eta\} \cap K_{\eta R}(z_o, 0)$. From properties (iii) and (iv) of v_η in Lemma 4.1 and Bishop's volume comparison theorem in Lemma 2.2, we deduce that

$$\begin{aligned} |K_{\alpha_1 R, \alpha_2 R^2}(z_o, 4R^2)|^{\frac{1}{n+1}} &\leq C_\eta \left\| \left(R^2 \mathcal{L} u \right)^+ \right\|_{L^{n+1}(K_{\beta_1 R, \beta_2 R^2}(z_o, 4R^2))} + C_\eta \left\| \chi_{E_2} \right\|_{L^{n+1}(K_{\beta_1 R, \beta_2 R^2}(z_o, 4R^2))} \\ &\leq C_\eta \|R^2 f^+\|_{L^{n+1}(K_{\alpha_1 R, \alpha_2 R^2}(z_o, 4R^2))} + C_\eta \left| \{u + v_\eta \leq M_\eta\} \cap K_{\eta R}(z_o, 0) \right|^{\frac{1}{n+1}}, \end{aligned}$$

where $C_\eta > 0$ depends only on n, λ and $\eta > 0$. We note that $\{u \leq M_\eta - v_\eta\} \subset \{u \leq M_\eta + C_\eta\}$ from (v) in Lemma 4.1. Therefore, by taking

$$\varepsilon_\eta = \frac{1}{2C_\eta}, \quad M'_\eta = M_\eta + C_\eta \quad \text{and} \quad \mu_\eta^{\frac{1}{n+1}} = \frac{1}{2C_\eta},$$

we conclude that $\frac{|\{u \leq M'_\eta\} \cap K_{\eta R}(z_o, 0)|}{|K_{\alpha_1 R, \alpha_2 R^2}(z_o, 4R^2)|} \geq \mu_\eta > 0$. \square

Using iteration of Lemma 4.3, we have the following corollaries.

Corollary 4.4. *Suppose that M satisfies the conditions (3),(4). Let $z_o \in M$ and $0 < \eta < 1$. For $i \in \mathbb{N}$, let $\bar{R}_i := \left(\frac{2}{\eta}\right)^{i-1} R$ and $\bar{t}_i := \sum_{j=1}^i 4\bar{R}_j^2$. Let u be a nonnegative smooth function such that $\mathcal{L}u \leq f$ in $\bigcup_{i=1}^k K_{\alpha_1 \bar{R}_i, \alpha_2 \bar{R}_i^2}(z_o, \bar{t}_i)$ for some $k \in \mathbb{N}$. We assume that for $h > 0$,*

$\inf_{\bigcup_{i=1}^k K_{2\bar{R}_i}(z_o, \bar{t}_i)} u \leq h$ and

$$\bar{R}_i^2 \left(\int_{K_{\alpha_1 \bar{R}_i, \alpha_2 \bar{R}_i^2}(z_o, \bar{t}_i)} |f^+|^{n+1} \right)^{\frac{1}{n+1}} \leq \varepsilon_\eta h M_\eta^{k-i}, \quad \forall 1 \leq i \leq k.$$

Then we have

$$(15) \quad \frac{|\{u \leq h M_\eta^k\} \cap K_{\eta R}(z_o, 0)|}{|K_{\alpha_1 R, \alpha_2 R^2}(z_o, 4R^2)|} \geq \mu_\eta,$$

where $M_\eta, \varepsilon_\eta, \mu_\eta$ are the same uniform constants as in Lemma 4.3.

Proof. We may assume $h = 1$ since $v := \frac{u}{h}$ satisfies $\mathcal{L}v = \frac{1}{h} \mathcal{L}u \leq \frac{f}{h}$. We use the induction on k to show the lemma. When $k = 1$, it is immediate from Lemma 4.3.

Now suppose that (15) is true for $k - 1$. By assumption, we find a $j_o \in \mathbb{N}$ such that $1 \leq j_o \leq k$ and $\inf_{K_{2\bar{R}_{j_o}}(z_o, \bar{t}_{j_o})} u = \inf_{\bigcup_{i=1}^k K_{2\bar{R}_i}(z_o, \bar{t}_i)} u \leq 1$. Define $v := u/M_\eta^{k-j_o}$. Then v satisfies that $\mathcal{L}v \leq f/M_\eta^{k-j_o}$, $\inf_{K_{2\bar{R}_{j_o}}(z_o, \bar{t}_{j_o})} v \leq 1$ and

$$\bar{R}_{j_o}^2 \left(\int_{K_{\alpha_1 \bar{R}_{j_o}, \alpha_2 \bar{R}_{j_o}^2}(z_o, \bar{t}_{j_o})} |f^+ / M_\eta^{k-j_o}|^{n+1} \right)^{\frac{1}{n+1}} \leq \varepsilon_\eta.$$

Applying Lemma 4.3 to v in $K_{\alpha_1 \bar{R}_{j_o}, \alpha_2 \bar{R}_{j_o}^2}(z_o, \bar{t}_{j_o})$, we deduce

$$\frac{|\{v \leq M_\eta\} \cap K_{\eta \bar{R}_{j_o}}(z_o, \bar{t}_{j_o} - 4R_{j_o}^2)|}{|K_{\alpha_1 \bar{R}_{j_o}, \alpha_2 \bar{R}_{j_o}^2}(z_o, \bar{t}_{j_o})|} = \frac{|\{v \leq M_\eta\} \cap K_{2\bar{R}_{j_o-1}}(z_o, \bar{t}_{j_o-1})|}{|K_{\alpha_1 \bar{R}_{j_o}, \alpha_2 \bar{R}_{j_o}^2}(z_o, \bar{t}_{j_o})|} \geq \mu_\eta > 0$$

which implies that $\inf_{\bigcup_{i=1}^{j_o-1} K_{2\bar{R}_i}(z_o, \bar{t}_i)} u \leq \inf_{K_{2\bar{R}_{j_o-1}}(z_o, \bar{t}_{j_o-1})} u \leq M_\eta^{k-j_o+1}$. Therefore, we use the induction hypothesis for $j_o - 1 (\leq k - 1)$ to conclude

$$\frac{|\{u/M_\eta^{k-j_o+1} \leq M_\eta^{j_o-1}\} \cap K_{\eta R}(z_o, 0)|}{|K_{\alpha_1 R, \alpha_2 R^2}(z_o, 4R^2)|} \geq \mu_\eta > 0,$$

which implies (15). \square

We remark that Lemma 4.3 and Corollary 4.4 hold for any $M'_\eta \geq M_\eta$. The following is a simple technical lemma that will be used in the proof of Proposition 4.6.

Lemma 4.5. *Let $A, D > 0$ and $\varepsilon > 0$. Let u be a nonnegative smooth function such that $\mathcal{L}u \leq f$ in $B_R(z_o) \times (-AR^2, 0]$ with*

$$R^2 \left(\int_{B_R(z_o) \times (-AR^2, 0]} |f^+|^{n+1} \right)^{\frac{1}{n+1}} \leq \varepsilon.$$

Then, there exists a sequence u_k of nonnegative smooth functions in $B_R(z_o) \times (-AR^2, DR^2]$ such that u_k converges to u locally uniformly in $B_R(z_o) \times (-AR^2, 0]$ and $\mathcal{L}u_k \leq g_k$ in $B_R(z_o) \times (-AR^2, DR^2]$ with

$$R^2 \left(\int_{B_R(z_o) \times (-AR^2, DR^2]} |g_k^+|^{n+1} \right)^{\frac{1}{n+1}} \leq \varepsilon.$$

Proof. First, we define for $(x, t) \in B_R(z_o) \times (-\infty, DR^2]$,

$$\bar{u}(x, t) := \begin{cases} 0 & \text{for } t \in (-\infty, -AR^2], \\ u(x, t) & \text{for } t \in (-AR^2, 0], \\ u(x, 0) + St & \text{for } t \in (0, DR^2], \end{cases}$$

where $S := \sup_{B_R(z_o)} \{(\mathcal{L}u)^+(x, 0) + |u_t(x, 0)|\}$. Then \bar{u} is Lipschitz continuous with respect to time in $B_R(z_o) \times (-AR^2, DR^2]$ and satisfies

$$\mathcal{L}\bar{u}(x, t) \leq \bar{f}(x, t) := \begin{cases} 0 & \text{for } t \in (-\infty, -AR^2), \\ f(x, t) & \text{for } t \in (-AR^2, 0), \\ \mathcal{L}u(x, 0) + u_t(x, 0) - S \leq 0 & \text{for } t \in (0, DR^2]. \end{cases}$$

Let $\varepsilon_k > 0$ converge to 0 as $k \rightarrow +\infty$, and let φ be a nonnegative smooth function such that $\varphi(t) = 0$ for $t \notin (0, 1)$ and $\int_{\mathbb{R}} \varphi(t) dt = 1$. We define $\varphi_k(t) := \frac{1}{\varepsilon_k} \varphi\left(\frac{t}{\varepsilon_k}\right)$ and

$$u_k(x, t) := \int_{\mathbb{R}} \bar{u}(x, s) \varphi_k(t - s) ds, \quad \forall (x, t) \in B_R(z_o) \times (-\infty, DR^2],$$

where we notice that the above integral is calculated over $(t - \varepsilon_k, t) \subset \mathbb{R}$. Then, a smooth function u_k satisfies

$$\mathcal{L}u_k(x, t) = \int_{\mathbb{R}} \mathcal{L}\bar{u}(x, s) \varphi_k(t - s) ds \leq g_k(x, t), \quad \forall (x, t) \in B_R(z_o) \times (-\infty, DR^2],$$

where $g_k(x, t) := \int_{\mathbb{R}} \bar{f}^+(x, s) \varphi_k(t - s) ds \geq 0$. We also have

$$\begin{aligned} R^2 \left(\int_{B_R(z_o) \times (-AR^2, DR^2]} |g_k^+|^{n+1} \right)^{\frac{1}{n+1}} &\leq \frac{R^2}{\{|B_R(z_o)| \cdot (A + D)R^2\}^{\frac{1}{n+1}}} \|\bar{f}^+\|_{L^{n+1}(B_R(z_o) \times (-AR^2 - \varepsilon_k, DR^2 - \varepsilon_k])} \\ &\leq \frac{R^2}{\{|B_R(z_o)| \cdot (A + D)R^2\}^{\frac{1}{n+1}}} \|\bar{f}^+\|_{L^{n+1}(B_R(z_o) \times (-AR^2, 0])} \\ &\leq \left(\frac{A}{A + D} \right)^{\frac{1}{n+1}} \varepsilon < \varepsilon, \end{aligned}$$

which finishes the proof. \square

Proposition 4.6. *Suppose that M satisfies the conditions (3),(4). Let $z_o \in M, R > 0, 0 < \eta < \frac{1}{2}$ and $\tau \in [3, 16]$. Let u be a nonnegative smooth function such that $\mathcal{L}u \leq f$ in $B_{\frac{49}{\eta}R}(z_o) \times (-3R^2, \frac{\tau R^2}{\eta^2}]$. Assume that $\inf_{B_R(z_o) \times [\frac{2R^2}{\eta^2}, \frac{\tau R^2}{\eta^2}]} u \leq 1$ and*

$$R^2 \left(\int_{B_{\frac{49}{\eta}R}(z_o) \times (-3R^2, \frac{\tau R^2}{\eta^2}]} |f^+|^{n+1} \right)^{\frac{1}{n+1}} \leq \varepsilon'_\eta$$

for a uniform constant $0 < \varepsilon'_\eta < 1$. Let $r > 0$ satisfy $\left(\frac{\eta}{2}\right)^N R \leq r < \left(\frac{\eta}{2}\right)^{N-1} R$ for some $N \in \mathbb{N}$ and let (z_1, t_1) be a point such that $d(z_o, z_1) < R$ and $|t_1| < R^2$. Then there exists a uniform constant $M'_\eta > 1$ (independent of r, N, z_1 and t_1) such that

$$\frac{|\{u \leq M'_\eta{}^{N+2}\} \cap K_{\eta r}(z_1, t_1)|}{|K_{\alpha_1 r, \alpha_2 r^2}(z_1, t_1 + 4r^2)|} \geq \mu_\eta > 0,$$

where $0 < \mu_\eta < 1$ is the constant in Lemma 4.3.

Proof. (i) From Lemma 4.5, we approximate u by nonnegative smooth functions u_k , which are defined on $B_{\frac{48}{\eta^3}R}(z_o) \times \left(-3R^2, \frac{64R^2}{(4-\eta^2)\eta^6} + R^2\right]$. We find functions u_k and g_k such that u_k converges locally uniformly to u in $B_{\frac{48}{\eta^3}R}(z_o) \times \left(-3R^2, \frac{\tau R^2}{\eta^2}\right]$, and satisfies

$$\mathcal{L}u_k \leq g_k \quad \text{in } B_{\frac{48}{\eta^3}R}(z_o) \times \left(-3R^2, \frac{64R^2}{(4-\eta^2)\eta^6} + R^2\right],$$

and

$$R^2 \left(\int_{B_{\frac{48}{\eta^3}R}(z_o) \times \left(-3R^2, \frac{64R^2}{(4-\eta^2)\eta^6} + R^2\right]} |g_k^+|^{n+1} \right)^{\frac{1}{n+1}} \leq \frac{49}{48} \varepsilon'_\eta < 2\varepsilon'_\eta$$

by using the volume comparison theorem and Lemma 4.5. For a small $\delta > 0$, we consider $w_k := \frac{u_k}{1+\delta}$ and then for large k , w_k satisfies $\inf_{B_R(z_o) \times \left[\frac{2R^2}{\eta^2}, \frac{\tau R^2}{\eta^2}\right]} w_k \leq 1$, $\mathcal{L}w_k \leq g_k$ in $B_{\frac{48}{\eta^3}R}(z_o) \times \left(-3R^2, \frac{64R^2}{(4-\eta^2)\eta^6} + R^2\right]$, and

$$R^2 \left(\int_{B_{\frac{48}{\eta^3}R}(z_o) \times \left(-3R^2, \frac{64R^2}{(4-\eta^2)\eta^6} + R^2\right]} |g_k^+|^{n+1} \right)^{\frac{1}{n+1}} < 2\varepsilon'_\eta,$$

according to the local uniform convergence of u_k to u in Lemma 4.5. So if we show the proposition for w_k , the local uniform convergence will imply that the result holds for u by letting $k \rightarrow +\infty$ and $\delta \rightarrow 0$. Now we assume that u is a nonnegative smooth function in $B_{\frac{48}{\eta^3}R}(z_o) \times \left(-3R^2, \frac{64R^2}{(4-\eta^2)\eta^6} + R^2\right]$ satisfying the same hypotheses as w_k .

(ii) We use Corollary 4.4 so we need to check the two hypotheses with $k = N+2$ and $h = 1$. As in the corollary, we define for $i \in \mathbb{N}$,

$$\bar{r}_i := \left(\frac{2}{\eta}\right)^{i-1} r \quad \text{and} \quad \bar{t}_i := t_1 + \sum_{j=1}^i 4\bar{r}_j^2.$$

Using the conditions on r , z_1 , and t_1 , simple computation says that for $0 < \eta < 1/2$,

$$B_{2\bar{r}_{N+1}}(z_1) \supset B_{2R}(z_1) \supset B_R(z_o),$$

$$\bar{t}_N < R^2 + \frac{16R^2}{4-\eta^2} < \frac{2R^2}{\eta^2} < \frac{16R^2}{\eta^2} < -R^2 + \frac{4(4+\eta^2)R^2}{\eta^2} < \bar{t}_{N+2}.$$

Thus we have $B_{2\bar{r}_{N+1}}(z_1) \times (\bar{t}_N, \bar{t}_{N+2}) \supset B_R(z_o) \times \left[\frac{2R^2}{\eta^2}, \frac{16R^2}{\eta^2}\right] \supset B_R(z_o) \times \left[\frac{2R^2}{\eta^2}, \frac{\tau R^2}{\eta^2}\right]$ for $0 < \eta < \frac{1}{2}$ and hence $\inf_{\bigcup_{i=1}^{N+2} K_{2\bar{r}_i}(z_1, \bar{t}_i)} u \leq \inf_{\bigcup_{i=N+1}^{N+2} K_{2\bar{r}_i}(z_1, \bar{t}_i)} u \leq 1$. We remark that \bar{r}_{N+2} is comparable to R .

Now, it suffices to show for some large $M'_\eta \geq M_\eta$, and small $0 < \varepsilon'_\eta < \varepsilon_\eta$, we have

$$(16) \quad \bar{r}_i^2 \left(\int_{K_{\alpha_1 \bar{r}_i, \alpha_2 \bar{r}_i^2}(z_1, \bar{t}_i)} |f^+|^{n+1} \right)^{\frac{1}{n+1}} \leq \varepsilon_\eta M'_\eta{}^{N+2-i}, \quad \forall 1 \leq i \leq N+2,$$

where M_η and ε_η are the constants in Corollary 4.4. We notice that $B_{\beta_1 \bar{r}_{N+2}}(z_o) \subset B_{\alpha_1 \bar{r}_{N+2}}(z_1) \subset B_{\frac{12}{\eta} \cdot \frac{4}{\eta^2} R}(z_o)$ and

$$\bigcup_{i=1}^{N+2} K_{\alpha_1 \bar{r}_i, \alpha_2 \bar{r}_i^2}(z_1, \bar{t}_i) \subset B_{\frac{48}{\eta^3}R}(z_o) \times \left(-3R^2, \frac{64R^2}{(4-\eta^2)\eta^6} + R^2\right]$$

since $d(z_o, z_1) < R$, $|t_1| < R^2$ and $\frac{2}{\eta}R \leq \bar{r}_{N+2} < \frac{4}{\eta}R$. Then for $i = 1, 2, \dots, N+2$, we have

$$\begin{aligned} \bar{r}_i^{2(n+1)} \int_{K_{\alpha_1 \bar{r}_i, \alpha_2 \bar{r}_i^2}(z_1, \bar{t}_i)} |f^+|^{n+1} &\leq \left(\frac{4}{\eta^2}\right)^{2(n+1)} \frac{R^{2(n+1)}}{|K_{\alpha_1 \bar{r}_i, \alpha_2 \bar{r}_i^2}(z_1, \bar{t}_i)|} \|f^+\|_{L^{n+1}\left(B_{\frac{48}{\eta^3}R}(z_o) \times \left(-3R^2, \frac{64R^2}{(4-\eta^2)\eta^6} + R^2\right)\right)}^{n+1} \\ &\leq \left(\frac{4}{\eta^2}\right)^{2(n+1)} (2\varepsilon'_\eta)^{n+1} \frac{\left|B_{\frac{48}{\eta^3}R}(z_o) \times \left(-3R^2, \frac{64R^2}{(4-\eta^2)\eta^6} + R^2\right)\right|}{|K_{\alpha_1 \bar{r}_i, \alpha_2 \bar{r}_i^2}(z_1, \bar{t}_i)|} \\ &\leq C(n, \eta) \varepsilon'_\eta^{n+1} \frac{|B_{\frac{48}{\eta^3}R}(z_o)| R^2}{|B_{\alpha_1 \bar{r}_i}(z_1)| \bar{r}_i^2} \leq C(n, \eta) \varepsilon'_\eta^{n+1} \frac{|B_{\beta_1 \bar{r}_{N+2}}(z_o)| \bar{r}_{N+2}^2}{|B_{\alpha_1 \bar{r}_i}(z_1)| \bar{r}_i^2}, \end{aligned}$$

where we use that $\frac{2}{\eta}R \leq \bar{r}_{N+2} < \frac{4}{\eta}R$ and the volume comparison theorem in the last inequality and the constant $C(n, \eta) > 0$ depending only on n and η , may change from line to line. Since $d(z_o, z_1) < R$, we use the volume comparison theorem again to obtain

$$\begin{aligned} \bar{r}_i^{2(n+1)} \int_{K_{\alpha_1 \bar{r}_i, \alpha_2 \bar{r}_i^2}(z_1, \bar{t}_i)} |f^+|^{n+1} &\leq C(n, \eta) \varepsilon'_\eta^{n+1} \frac{|B_{\beta_1 \bar{r}_{N+2}}(z_o)| \bar{r}_{N+2}^2}{|B_{\alpha_1 \bar{r}_i}(z_1)| \bar{r}_i^2} \\ &\leq C(n, \eta) \varepsilon'_\eta^{n+1} \frac{|B_{\alpha_1 \bar{r}_{N+2}}(z_1)| \bar{r}_{N+2}^2}{|B_{\alpha_1 \bar{r}_i}(z_1)| \bar{r}_i^2} \\ &\leq C(n, \eta) \varepsilon'_\eta^{n+1} \left(\frac{\bar{r}_{N+2}}{\bar{r}_i}\right)^{n+2} \leq C(n, \eta) \varepsilon'_\eta^{n+1} \left(\frac{2}{\eta}\right)^{(n+2)(N+2-i)}. \end{aligned}$$

We select $M'_\eta > M_\eta$ large and $0 < \varepsilon'_\eta < \varepsilon_\eta$ small enough to satisfy

$$C(n, \eta) \varepsilon'_\eta^{n+1} \left(\frac{2}{\eta}\right)^{(n+2)(N+2-i)} \leq \varepsilon_\eta^{n+1} M_\eta'^{(n+1)(N+2-i)}, \quad \forall 1 \leq i \leq N+2,$$

which proves (16). Therefore, Corollary 4.4 (after translation in time by t_1) gives

$$\frac{\left|\{u \leq M_\eta'^{N+2}\} \cap K_{\eta r}(z_1, t_1)\right|}{|K_{\alpha_1 r, \alpha_2 r^2}(z_1, t_1 + 4r^2)|} \geq \mu_\eta > 0.$$

□

5. PARABOLIC VERSION OF THE CALDERÓN-ZYGMUND DECOMPOSITION

Throughout this section, we assume that a complete Riemannian manifold M satisfies the condition (3). We introduce a parabolic version of the Calderón-Zygmund lemma (Lemma 5.7) to prove power decay of super-level sets in Lemma 6.1 (see [W, Ca, CC]). Christ [Ch] proved that the following theorem holds for so-called "spaces of homogeneous type", which is a generalization of Euclidean dyadic decomposition. In harmonic analysis, a metric space X is called a space of homogeneous type when X equips a nonnegative Borel measure ν satisfying the doubling property

$$\nu(B_{2R}(x)) \leq A_1 \nu(B_R(x)) < +\infty, \quad \forall x \in X, R > 0,$$

for some constant A_1 independent of x and R . From Bishop's volume comparison (Lemma 2.2), a complete Riemannian manifold M satisfying the condition (3) is a space of homogeneous type with $A_1 = 2^n$.

Theorem 5.1 (Christ). *There exist a countable collection $\{Q^{k,\alpha} \subset M : k \in \mathbb{Z}, \alpha \in I_k\}$ of open subsets of M and positive constants $0 < \delta_0 < 1$, c_1 and c_2 (with $2c_1 \leq c_2$) that depend only on n , such that*

- (i) $|M \setminus \bigcup_\alpha Q^{k,\alpha}| = 0$ for $k \in \mathbb{Z}$,
- (ii) if $l \leq k$, $\alpha \in I_k$, and $\beta \in I_l$, then either $Q^{k,\alpha} \subset Q^{l,\beta}$ or $Q^{k,\alpha} \cap Q^{l,\beta} = \emptyset$,
- (iii) for any (k, α) and any $l < k$, there is a unique β such that $Q^{k,\alpha} \subset Q^{l,\beta}$,
- (iv) $\text{diam}(Q^{k,\alpha}) \leq c_2 \delta_0^k$,
- (v) any $Q^{k,\alpha}$ contains some ball $B_{c_1 \delta_0^k}(z^{k,\alpha})$.

For convenience, we will use the following notation.

Definition 5.2 (Dyadic cubes on M).

- (i) The open set $Q = Q^{k,\alpha}$ in Theorem 5.1 is called a dyadic cube of generation k on M . From the property (iii) in Theorem 5.1, for any (k, α) , there is a unique β such that $Q^{k,\alpha} \subset Q^{k-1,\beta}$. We call $Q^{k-1,\beta}$ the predecessor of $Q^{k,\alpha}$. When $Q := Q^{k,\alpha}$, we denote the predecessor $Q^{k-1,\beta}$ by \tilde{Q} for simplicity.
- (ii) For a given $R > 0$, we define $k_R \in \mathbb{N}$ to satisfy

$$c_2 \delta_0^{k_R-1} < R \leq c_2 \delta_0^{k_R-2}.$$

The number k_R means that a dyadic cube of generation k_R is comparable to a ball of radius R .

For the rest of the paper, we fix some small numbers;

$$\delta := \frac{2c_1}{c_2} \delta_0 \in (0, \delta_0), \quad \delta_1 := \frac{\delta_0(1-\delta_0)}{2} \in \left(0, \frac{\delta_0}{2}\right),$$

$$\eta := \min(\delta, \delta_1) \in \left(0, \frac{1}{2}\right) \quad \text{and} \quad \kappa := \frac{\eta}{2} \sqrt{1 - \delta_0^2}.$$

By using the dyadic decomposition of a manifold M , we have the following decomposition of $M \times (T_1, T_2]$ in space and time. For time variable, we take the standard euclidean dyadic decomposition.

Lemma 5.3. *There exists a countable collection $\{K^{k,\alpha} \subset M \times (T_1, T_2] : k \in \mathbb{Z}, \alpha \in J_k\}$ of subsets of $M \times (T_1, T_2] \subset M \times \mathbb{R}$ and positive constants $0 < \delta_0 < 1$, c_1 and c_2 (with $2c_1 \leq c_2$) that depend only on n , such that*

- (i) $|M \times (T_1, T_2] \setminus \bigcup_\alpha K^{k,\alpha}| = 0$ for $k \in \mathbb{Z}$,
- (ii) if $l \leq k$, $\alpha \in J_k$, and $\beta \in J_l$, then either $K^{k,\alpha} \subset K^{l,\beta}$ or $K^{k,\alpha} \cap K^{l,\beta} = \emptyset$,
- (iii) for any (k, α) and any $l < k$, there is a unique β such that $K^{k,\alpha} \subset K^{l,\beta}$,
- (iv) $\text{diam}(K^{k,\alpha}) \leq c_2 \delta_0^k \times c_2^2 \delta_0^{2k}$,
- (v) any $K^{k,\alpha}$ contains some cylinder $B_{c_1 \delta_0^k}(z^{k,\alpha}) \times (t^{k,\alpha} - c_1^2 \delta_0^{2k}, t^{k,\alpha}]$.

Proof. To decompose in time variable, for each $k \in \mathbb{Z}$, we select the largest integer $N_k \in \mathbb{Z}$ to satisfy

$$\frac{1}{4} c_2^2 \delta_0^{2k} \leq \frac{T_2 - T_1}{2^{2N_k}} < c_2^2 \delta_0^{2k}.$$

For k -th generation, we split the interval $(T_1, T_2]$ into 2^{2N_k} disjoint subintervals which have the same length. Then we obtain $|J_k| = |I_k| \cdot 2^{2N_k}$ disjoint subsets on $M \times (T_1, T_2]$ satisfying properties (i)-(v). \square

For the rest of this section, let $\{K^{k,\alpha} \subset M \times (T_1, T_2] : k \in \mathbb{Z}, \alpha \in J_k\}$ be the parabolic dyadic decomposition of $M \times (T_1, T_2]$ as in Lemma 5.3.

Definition 5.4 (Parabolic dyadic cubes).

- (i) $K = K^{k,\alpha}$ is called a parabolic dyadic cube of generation k . If $K := K^{k,\alpha} \subset K^{k-1,\beta} =: \widetilde{K}$, we say \widetilde{K} is the predecessor of K .
- (ii) For a parabolic dyadic cube K of generation k , we define $l(k)$ to be the length of K in time variable, namely, $l(k) = \frac{T_2 - T_1}{2^{2k}}$ for $M \times (T_1, T_2]$ in Lemma 5.3.

We quote the following technical lemma proven by Cabré [Ca, Lemma 6.5].

Lemma 5.5 (Cabré). *Let $z_o \in M$ and $R > 0$. Then we have the following.*

- (i) *If Q is a dyadic cube of generation k such that*

$$k \geq k_R \quad \text{and} \quad Q \subset B_R(z_o),$$

then there exist $z_1 \in Q$ and $r_k \in (0, R/2)$ such that

$$(17) \quad B_{\delta r_k}(z_1) \subset Q \subset \widetilde{Q} \subset \overline{B_{2r_k}(z_1)} \subset B_{\frac{1}{\eta} r_k}(z_1) \subset B_{\frac{1}{\eta} R}(z_o)$$

and

$$(18) \quad B_{\frac{1}{\eta} R}(z_o) \subset B_{\frac{1}{\eta} R}(z_1).$$

In fact, for $k \geq k_R$, the above radius r_k is defined by

$$r_k := \frac{1}{2} c_2 \delta_0^{k-1} = \frac{c_1}{\delta} \delta_0^k.$$

- (ii) *If Q is a dyadic cube of generation k_R and $d(z_o, Q) \leq \delta_1 R$, then $Q \subset B_R(z_o)$ and hence (17) and (18) hold for some $z_1 \in Q$ and $r_{k_R} \in \left[\frac{\delta_0 R}{2}, \frac{R}{2}\right)$. Moreover,*

$$B_{\delta_1 R}(z_o) \subset B_{2r_{k_R}}(z_1).$$

- (iii) *There exists at least one dyadic cube Q of generation k_R such that $d(z_o, Q) \leq \delta_1 R$.*

We remark that for $k \geq k_R$,

$$\eta^2 r_k^2 \leq \delta_0^2 r_k^2 = \frac{1}{4} c_2^2 \delta_0^{2k} \leq l(k) < c_2^2 \delta_0^{2k} = 4r_{k+1}^2$$

and (17) gives that for any $a \in \mathbb{R}$,

$$(19) \quad K_{\eta r_k}(z_1, a) \subset Q \times (a - l(k), a] \subset \overline{K_{2r_k}(z_1, a)}$$

Definition 5.6. *Let $m \in \mathbb{N}$. For any parabolic dyadic cube $K := Q \times (a - l(k), a]$ of generation k , the elongation of K along time in m steps (see [KL]), denoted by \overline{K}^m , is defined by*

$$\overline{K}^m := \widetilde{Q} \times (a, a + m \cdot l(k - 1)],$$

where $l(k)$ is the length of a parabolic dyadic cube of generation k in time and \widetilde{Q} is the predecessor of Q in space. The elongation \overline{K}^m is the union of the stacks of parabolic dyadic cubes congruent to the predecessor of K .

Now we have a parabolic version of Calderón-Zygmund lemma. The proof of lemma is the same as Euclidean case so we refer to [W] for the proof.

Lemma 5.7 (Lemma 3.23, [W]). *Let $K_1 = Q_1 \times (a - l(k_0), a]$ be a parabolic dyadic cube of generation k_0 in $M \times (T_1, T_2]$, and let $0 < \alpha < 1$ and $m \in \mathbb{N}$. Let $\mathcal{A} \subset K_1$ be a measurable set such that $|\mathcal{A} \cap K_1| \leq \alpha |K_1|$ and let*

$$\mathcal{A}_\alpha^m := \cup \left\{ \overline{K}^m : |K \cap \mathcal{A}| > \alpha |K|, K, \text{ a parabolic dyadic cube in } K_1 \right\} \cap (Q_1 \times \mathbb{R}).$$

Then, we have

$$|\mathcal{A}_\alpha^m| \geq \frac{m}{(m+1)\alpha} |\mathcal{A}|.$$

6. HARNACK INEQUALITY

In order to prove the parabolic Harnack inequality, we take the approach presented in [W] and iterate Lemma 4.3 with Christ decomposition (Theorem 5.1) and Calderón-Zygmund type lemma (Lemma 5.7). We begin this section with recalling that $\eta \in (0, \frac{1}{2})$ is fixed as in the previous section. So the uniform constants $\mu_\eta, \varepsilon'_\eta$ and M'_η in Proposition 4.6 are also fixed and we denote them by μ, ε_0 and M_0 for simplicity.

We select an integer $m > 1$ large enough to satisfy

$$\frac{m}{(m+1)(1-\mu)} > \frac{1}{1-\frac{\mu}{2}},$$

where $0 < \mu < 1$ is the constant in Lemma 4.3. For $T_1 := -3R^2$ and $T_2 := (\frac{16}{\eta^2} + 1 + m)R^2$, we consider a parabolic dyadic decomposition of $M \times (T_1, T_2]$ in Lemma 5.3 and fix the decomposition for Section 6.

6.1. Power decay estimate of super-level sets.

Lemma 6.1. *Suppose that M satisfies the conditions (3),(4). Let $z_o \in M, R > 0$ and $\tau \in [3, 16]$. Let u be a nonnegative smooth function such that $\mathcal{L}u \leq f$ in $B_{\frac{50}{\eta^3}R}(z_o) \times (-3R^2, \frac{\tau R^2}{\eta^2}]$. Assume that*

$$\inf_{B_R(z_o) \times [\frac{2R^2}{\eta^2}, \frac{\tau R^2}{\eta^2}]} u \leq 1$$

and

$$R^2 \left(\int_{B_{\frac{50}{\eta^3}R}(z_o) \times (-3R^2, \frac{\tau R^2}{\eta^2}]} |f^+|^{n+1} \right)^{\frac{1}{n+1}} \leq \varepsilon_1$$

for a uniform constant $0 < \varepsilon_1 < \varepsilon_0$. Let K_1 be a parabolic dyadic cube of generation k_R such that

$$K_1 := Q_1 \times (t_1 - l(k_R), t_1] \subset Q_1 \times (-R^2, R^2),$$

where Q_1 is a dyadic cube of generation k_R such that $d(z_o, Q_1) \leq \delta_1 R$. Then for $i = 1, 2, \dots$, we have

$$(20) \quad \frac{|\{u > M_1^i\} \cap K_1|}{|K_1|} < \left(1 - \frac{\mu}{2}\right)^i,$$

where $0 < \varepsilon_1 < \varepsilon_0$ and $M_1 > 0$ depend only on n, λ, Λ , and a_L .

Proof. (i) As Proposition 4.6, we use Lemma 4.5 to assume that a nonnegative smooth function u defined on $B_{\frac{49}{\eta^3}R}(z_o) \times (T_1, T_2]$ satisfies that $\inf_{B_R(z_o) \times [\frac{2R^2}{\eta^2}, \frac{\tau R^2}{\eta^2}]} u \leq 1$ and $\mathcal{L}u \leq f$ in

$B_{\frac{49}{\eta^3}R}(z_o) \times (T_1, T_2]$ for some f with

$$R^2 \left(\int_{B_{\frac{49}{\eta^3}R}(z_o) \times (T_1, T_2]} |f^+|^{n+1} \right)^{\frac{1}{n+1}} \leq \frac{50}{49} \varepsilon_1 < 2\varepsilon_1.$$

(ii) According to Lemma 5.5, there exists a dyadic cube $Q_1 \subset B_R(z_o)$ of generation k_R such that $d(z_o, Q_1) \leq \delta_1 R$. We find $z_1 \in Q_1$ and $r_{k_R} \in [\frac{\delta_0}{2}R, \frac{1}{2}R]$ satisfying (17),(18) and $B_{\delta_1 R}(z_o) \subset B_{2r_{k_R}}(z_1)$. Since $\eta^2 r_{k_R}^2 \leq l(k_R) < 4r_{k_R+1}^2 = 4\delta_0^2 r_{k_R}^2 < \delta_0^2 R^2$, we find $t_1 \in$

$(-R^2 + l(k_R), R^2)$ such that $K_1 := Q_1 \times (t_1 - l(k_R), t_1]$ is a parabolic dyadic cube of generation k_R of $M \times (T_1, T_2]$. From (19), we also have that

$$K_{\eta r_{k_R}}(z_1, t_1) \subset K_1 \subset \overline{K_{2r_{k_R}}(z_1, t_1)}.$$

We use the induction to prove (20) so we first check the case $i = 1$. We notice that $d(z_o, z_1) < R$, $r_{k_R} \in [\frac{\delta_0}{2}R, \frac{1}{2}R) \subset (\frac{\eta}{2}R, R)$ and $|t_1| < R^2$. We set $\varepsilon_1 := \left(\frac{3/\eta^2+3}{16/\eta^2+m+4}\right)^{\frac{1}{n+1}} \frac{\varepsilon_0}{2}$. Then, u satisfies the hypotheses of Proposition 4.6 with $r = r_{k_R}$ and $N = 1$, so we deduce that

$$0 < \mu \leq \frac{|\{u \leq M_0^3\} \cap K_{\eta r_{k_R}}(z_1, t_1)|}{|K_{\alpha_1 r_{k_R}, \alpha_2 r_{k_R}^2}(z_1, t_1 + 4r_{k_R}^2)|} = \frac{|\{u \leq M_0^3\} \cap K_{\eta r_{k_R}}(z_1, t_1)|}{|K_{\alpha_1 r_{k_R}, \alpha_2 r_{k_R}^2}(z_1, t_1)|} < \frac{|\{u \leq M_0^3\} \cap K_1|}{|K_1|}.$$

Thus, we have for $M_1 \geq M_0^3$,

$$\frac{|\{u > M_1\} \cap K_1|}{|K_1|} \leq 1 - \mu < 1 - \frac{\mu}{2}.$$

(iii) Now, suppose that (20) is true for i , that is,

$$\frac{|\{u > M_1^i\} \cap K_1|}{|K_1|} < \left(1 - \frac{\mu}{2}\right)^i.$$

To show the $(i+1)$ -th step, define for $h > 0$,

$$\mathcal{B}_h := \{u > h\} \cap B_{\frac{49}{\eta^3}R}(z_o) \times (T_1, T_2].$$

We know $\frac{|\mathcal{B}_{M_1^i} \cap K_1|}{|K_1|} < \left(1 - \frac{\mu}{2}\right)^i$. If $h > 0$ is a constant such that

$$\frac{|\mathcal{A}|}{|K_1|} \geq \left(1 - \frac{\mu}{2}\right)^{i+1} \quad \text{for } \mathcal{A} := \mathcal{B}_{hM_1^i} \cap K_1,$$

then we will show that $h < M_1$ for a uniform constant $M_1 > M_0 > 1$, that will be fixed later.

Suppose on the contrary that $h \geq M_1$. From (ii), we have $\frac{|\mathcal{A}|}{|K_1|} \leq \frac{|\mathcal{B}_{M_0^3} \cap K_1|}{|K_1|} \leq 1 - \mu$ for $M_1 \geq M_0^3$ and $h \geq 1$. Applying Lemma 5.7 to \mathcal{A} with $\alpha = 1 - \mu$, it follows that

$$|\mathcal{A}_{1-\mu}^m| \geq \frac{m}{(m+1)(1-\mu)} |\mathcal{A}| > \frac{1}{1-\frac{\mu}{2}} |\mathcal{A}|.$$

We claim that

$$(21) \quad \mathcal{A}_{1-\mu}^m \subset \mathcal{B}_{\frac{hM_1^i}{M_0^m}}$$

for $h \geq C_1 M_0^m > 1$, where a uniform constant $C_1 > 0$ will be chosen. If not, there is a point $(x_1, s_1) \in \mathcal{A}_{1-\mu}^m \setminus \mathcal{B}_{\frac{hM_1^i}{M_0^m}}$ and we find a parabolic dyadic cube $K := Q \times (a - l(k), a] \subset K_1$ of generation $k(> k_R)$ such that

$$|\mathcal{A} \cap K| > (1 - \mu)|K| \quad \text{and} \quad (x_1, s_1) \in \overline{K}^m$$

from the definition of $\mathcal{A}_{1-\mu}^m$. According to Lemma 5.5, there exist $z_1 \in Q \subset Q_1 \subset B_R(z_o)$ and $r_k \in (0, R/2)$ satisfying (17), (18), $K_{\eta r_k}(z_1, a) \subset K \subset \overline{K_{2r_k}(z_1, a)}$ and

$$(x_1, s_1) \in \overline{K}^m = \widetilde{Q} \times (a, a + m \cdot l(k-1)] \subset \overline{B_{2r_k}(z_1)} \times (a, a + m \cdot 4r_k^2].$$

We note that

$$\inf_{B_{2r_k}(z_1) \times (a, a+m \cdot 4r_k^2]} u \leq u(x_1, s_1) \leq \frac{hM_1^i}{M_0^m}$$

and

$$B_{\alpha_1 r_k}(z_1) \times \left(a - (\eta^2 + \eta^4/4)r_k^2, a + m \cdot 4r_k^2 \right] \subset B_{\alpha_1 R}(z_o) \times (-3R^2, (1+m)R^2],$$

since $r_k < R/2$ and $a \in (t_1 - l(k_R), t_1] \subset (-R^2, R^2)$. We also have that for $j = 1, \dots, m$,

$$(22) \quad r_k^2 \left(\int_{K_{\alpha_1 r_k, \alpha_2 r_k^2}(z_1, a+(m-j+1) \cdot 4r_k^2)} |f^+|^{n+1} \right)^{\frac{1}{n+1}} \leq \varepsilon_0 \frac{hM_1^i}{M_0^{m-j+1}}.$$

Indeed, the volume comparison theorem and the property (18) will give that

$$\begin{aligned} r_k^2 \left(\int_{K_{\alpha_1 r_k, \alpha_2 r_k^2}(z_1, a+(m-j+1) \cdot 4r_k^2)} |f^+|^{n+1} \right)^{\frac{1}{n+1}} &= \frac{r_k^{\frac{n}{n+1}} \cdot r_k^{\frac{n+2}{n+1}}}{|B_{\alpha_1 r_k}(z_1)|^{\frac{1}{n+1}} (\alpha_2 r_k^2)^{\frac{1}{n+1}}} \|f^+\|_{L^{n+1}} \\ &\leq \frac{R^{\frac{n}{n+1}} \cdot R^{\frac{n+2}{n+1}}}{|B_{\alpha_1 R}(z_1)|^{\frac{1}{n+1}} (\alpha_2 R^2)^{\frac{1}{n+1}}} \|f^+\|_{L^{n+1}(B_{\frac{12}{\eta} R}(z_o) \times (T_1, T_2])} \\ &\leq \frac{R^2}{|B_{\beta_1 R}(z_o)|^{\frac{1}{n+1}} (\alpha_2 R^2)^{\frac{1}{n+1}}} \|f^+\|_{L^{n+1}(B_{\frac{49}{\eta^3}}(z_o) \times (T_1, T_2])} \\ &\leq \frac{C_1 R^2/2}{|B_{\frac{49}{\eta^3}}(z_o) \times (T_1, T_2)|^{\frac{1}{n+1}}} \|f^+\|_{L^{n+1}(B_{\frac{49}{\eta^3}}(z_o) \times (T_1, T_2])} < C_1 \varepsilon_1, \end{aligned}$$

where a uniform constant $C_1 > 1$ depends only on η, n and m . For $h \geq C_1 M_0^m$ and $M_1 > 1$, we have that

$$r_k^2 \left(\int_{K_{\alpha_1 r_k, \alpha_2 r_k^2}(z_1, a+(m-j+1) \cdot 4r_k^2)} |f^+|^{n+1} \right)^{\frac{1}{n+1}} < C_1 \varepsilon_1 < \varepsilon_0 \frac{hM_1^i}{M_0^m} \leq \varepsilon_0 \frac{hM_1^i}{M_0^{m-j+1}},$$

which proves (22). Thus, we can apply Lemma 4.3 iteratively to $\tilde{u}_j := \frac{M_0^{m-j+1}}{hM_1^i} u$, for $1 \leq j \leq m$, to deduce

$$\mu \leq \frac{\left| \left\{ u \leq hM_1^i \right\} \cap K_{\eta r_k}(z_1, a) \right|}{|K_{\alpha_1 r_k, \alpha_2 r_k^2}(z_1, a + 4r_k^2)|} < \frac{\left| \left\{ u \leq hM_1^i \right\} \cap K \right|}{|K|}.$$

However, this contradicts to the fact that $|\mathcal{A} \cap K| > (1 - \mu)|K|$. Therefore, we have proved that $\mathcal{A}_{1-\mu}^m \subset \mathcal{B}_{\frac{hM_1^i}{M_0^m}}$ for $h \geq C_1 M_0^m$.

(iv) Since $|\mathcal{B}_{M_1^i} \cap K_1| < \left(1 - \frac{\mu}{2}\right)^i |K_1|$, we have that $|\mathcal{B}_{\frac{hM_1^i}{M_0^m}} \cap K_1| \leq |\mathcal{B}_{M_1^i} \cap K_1| < \left(1 - \frac{\mu}{2}\right)^i |K_1| \leq \frac{1}{1 - \frac{\mu}{2}} |\mathcal{A}|$ for $h \geq C_1 M_0^m$. Then, by using (21), we obtain

$$\begin{aligned} |\mathcal{A}_{1-\mu}^m \setminus K_1| &= |\mathcal{A}_{1-\mu}^m| - |\mathcal{A}_{1-\mu}^m \cap K_1| \\ &\geq \frac{m}{(m+1)(1-\mu)} |\mathcal{A}| - \left| \mathcal{B}_{\frac{hM_1^i}{M_0^m}} \cap K_1 \right| \\ &> \left(\frac{m}{(m+1)(1-\mu)} - \frac{1}{1 - \frac{\mu}{2}} \right) |\mathcal{A}| =: \alpha |\mathcal{A}| \geq \alpha \left(1 - \frac{\mu}{2}\right)^{i+1} |K_1| \end{aligned}$$

with $\alpha := \frac{m}{(m+1)(1-\mu)} - \frac{1}{1 - \frac{\mu}{2}} > 0$. We find a point $(x_1, s_1) \in \mathcal{A}_{1-\mu}^m \setminus K_1$ and a parabolic dyadic cube $K := Q \times (a - l(k), a] \subset K_1$ of generation $k (> k_R)$ such that $(x_1, s_1) \in \bar{K}^m$, and $|\mathcal{A} \cap K| > (1 - \mu)|K|$. We may assume that

$$s_1 > t_1 + \frac{\alpha}{2} \left(1 - \frac{\mu}{2}\right)^{i+1} l(k_R)$$

since $\mathcal{A}_{1-\mu}^m \subset Q_1 \times (t_1 - l(k_R), +\infty)$ and $\frac{|\mathcal{A}_{1-\mu}^m \setminus K_1|}{|Q_1|} > \alpha \left(1 - \frac{\mu}{2}\right)^{i+1} l(k_R)$. Using Lemma 5.5 again, there exist $z_1 \in Q \subset Q_1 \subset B_R(z_o)$ and $r_k \in (0, R/2)$ satisfying (17), (18), and $K_{\eta r_k}(z_1, a) \subset K \subset \bar{K}_{2r_k}(z_1, a)$. Then we have

$$s_1 \leq a + m \cdot l(k - 1) < t_1 + m \cdot 4r_k^2$$

and hence

$$r_k \geq \frac{\sqrt{\alpha}}{\sqrt{8m}} \left(1 - \frac{\mu}{2}\right)^{\frac{i+1}{2}} \sqrt{l(k_R)} \geq \frac{\sqrt{\alpha} \delta_0^2}{4\sqrt{2m}} \left(1 - \frac{\mu}{2}\right)^{\frac{i+1}{2}} R \geq \left(\frac{\eta}{2}\right)^{Ni} R$$

for a uniform integer $N > 0$ independent of $i \in \mathbb{N}$. We apply Proposition 4.6 to u in order to get

$$\mu \leq \frac{|\{u \leq M_0^{Ni+2}\} \cap K_{\eta r_k}(z_1, a)|}{|K_{\alpha_1 r_k, \alpha_2 r_k^2}(z_1, a + 4r_k^2)|} \leq \frac{|\{u \leq M_0^{(N+2)i}\} \cap K|}{|K|},$$

since $r_k \geq \left(\frac{\eta}{2}\right)^{Ni} R$, and $(z_1, a) \in K \subset K_1 \subset B_R(z_o) \times (-R^2, R^2)$. If $h \geq M_1 := \max\{C_1 M_0^m, M_0^{N+2}\}$, this implies

$$1 - \mu > \frac{|\{u > M_0^{(N+2)i}\} \cap K|}{|K|} \geq \frac{|\{u > hM_1^i\} \cap K|}{|K|} = \frac{|\mathcal{A} \cap K|}{|K|},$$

which is a contradiction to the fact that $|\mathcal{A} \cap K| > (1 - \mu)|K|$. Thus, we have $h < M_1$ for a uniform constant $M_1 := \max\{C_1 M_0^m, M_0^{N+2}\}$. Therefore, we conclude that $\frac{|\{u > M_1^{i+1}\} \cap K_1|}{|K_1|} < \left(1 - \frac{\mu}{2}\right)^{i+1}$, completing the proof. \square

The following corollary is a direct consequence of Lemma 6.1, which estimates the distribution function of u .

Corollary 6.2. *Under the same assumption as Lemma 6.1, we have*

$$(23) \quad \frac{|\{u \geq h\} \cap K_1|}{|K_1|} \leq dh^{-\epsilon} \quad \forall h > 0,$$

where $d > 0$ and $0 < \epsilon < 1$ depend only on n, λ, Λ , and a_L .

Another consequence of Lemma 6.1 is a weak Harnack inequality for nonnegative supersolutions to $\mathcal{L}u = f$.

Corollary 6.3. *Under the same assumption as Lemma 6.1, we have for $p_o := \frac{\epsilon}{2}$,*

$$(24) \quad \left(\frac{1}{|K_{\kappa R}(z_o, 0)|} \int_{K_{\kappa R}(z_o, 0)} u^{p_o} \right)^{\frac{1}{p_o}} \leq C,$$

where $C > 0$ depends only on n, λ, Λ , and a_L .

Proof. Let $k = k_R$ and let $\{K^{k, \alpha} := Q^{k, \alpha} \times (t^{k, \alpha} - l(k), t^{k, \alpha}]\}_{\alpha \in J'_k}$ be a family of parabolic dyadic cubes intersecting $K_{\kappa R}(z_o, 0)$. For $\alpha \in J'_k$, we have that $K^{k, \alpha} \subset B_{(\kappa + \delta_0)R}(z_o) \times (-R^2, R^2]$ since $d(z_o, Q^{k, \alpha}) \leq \kappa R (< \delta_1 R)$, $\text{diam}(Q^{k, \alpha}) \leq c_2 \delta_0^k \leq \delta_0 R$, and $-R^2 + l(k) < -\kappa^2 R^2 \leq t^{k, \alpha} \leq l(k) < \delta_0^2 R^2$. Since

$$\begin{aligned} |K_{\kappa R}(z_o, 0)| &\geq \left(\frac{\kappa}{\kappa + \delta_0} \right)^n |B_{(\kappa + \delta_0)R}(z_o)| \cdot \kappa^2 R^2 \geq \left(\frac{\kappa}{\kappa + \delta_0} \right)^n \sum_{\alpha \in J'_k} |Q^{k, \alpha}| \cdot \kappa^2 R^2 \\ &\geq \left(\frac{\kappa}{\kappa + \delta_0} \right)^n \sum_{\alpha \in J'_k} |B_{c_1 \delta_0^k}(z^{k, \alpha})| \cdot \kappa^2 R^2 \geq \left(\frac{\kappa}{\kappa + \delta_0} \right)^n \sum_{\alpha \in J'_k} |B_{\frac{\delta \delta_0}{2} R}(z^{k, \alpha})| \cdot \kappa^2 R^2 \\ &\geq \left(\frac{\kappa}{\kappa + \delta_0} \cdot \frac{\delta \delta_0}{2(\delta_0 + 2\kappa)} \right)^n \sum_{\alpha \in J'_k} |B_{(\delta_0 + 2\kappa)R}(z^{k, \alpha})| \cdot \kappa^2 R^2 \\ &\geq \left(\frac{\kappa}{\kappa + \delta_0} \cdot \frac{\delta \delta_0}{2(\delta_0 + 2\kappa)} \right)^n \sum_{\alpha \in J'_k} |B_{\kappa R}(z_o)| \cdot \kappa^2 R^2, \end{aligned}$$

the number $|J'_k|$ of parabolic dyadic cubes intersecting $K_{\kappa R}(z_o, 0)$ is uniformly bounded. Thus for some $K^{k, \alpha}$ with $\alpha \in J'_k$, we have

$$\begin{aligned} \int_{K_{\kappa R}(z_o)} u^{p_o} &\leq |J'_k| \cdot \int_{K^{k, \alpha}} u^{p_o} \\ &\leq |J'_k| \cdot \left\{ |K^{k, \alpha}| + p_o \int_1^\infty h^{p_o-1} |\{u \geq h\} \cap K^{k, \alpha}| dh \right\} \\ &\leq |J'_k| \cdot \left\{ |K^{k, \alpha}| + p_o d |K^{k, \alpha}| \int_1^\infty h^{p_o-1-\epsilon} dh \right\}. \end{aligned}$$

from Corollary 6.2, where d and ϵ are the constants in Corollary 6.2.

By using the volume comparison theorem, we conclude that

$$\frac{1}{|K_{\kappa R}(z_o, 0)|} \int_{K_{\kappa R}(z_o, 0)} u^{p_o} \leq C_0 \frac{|K^{k, \alpha}|}{|K_{\kappa R}(z_o, 0)|} \leq C_0 \left(\frac{\kappa + \delta_0}{\kappa} \right)^n \cdot \frac{\delta_0^2}{\kappa^2}$$

for $C_0 := |J'_k| \cdot \left\{ 1 + p_o d \int_1^\infty h^{-1-\epsilon/2} dh \right\}$ since $K^{k, \alpha} \subset B_{(\kappa + \delta_0)R}(z_o) \times (t^{k, \alpha} - \delta_0^2 R^2, t^{k, \alpha}]$. \square

6.2. Proof of Harnack Inequality. So far, we have dealt with nonnegative supersolutions. Now, we consider a nonnegative solution u of $\mathcal{L}u = f$. We apply Corollary 6.2 as in [Ca] (see also [W]) to solutions $C_1 - C_2 u$ for some constants C_1 and C_2 .

Lemma 6.4. *Suppose that M satisfies the conditions (3), (4). Let $z_o \in M, R > 0$ and $\tau \in [3, 16]$. Let u be a nonnegative smooth function such that $\mathcal{L}u = f$ in $B_{\frac{50}{\eta} R}(z_o) \times (-3R^2, \frac{\tau R^2}{\eta^2}]$.*

Assume that $\inf_{B_R(z_o) \times \left[\frac{2R^2}{\eta^2}, \frac{\tau R^2}{\eta^2}\right]} u \leq 1$ and

$$R^2 \left(\int_{B_{\frac{50}{\eta^3}R}(z_o) \times \left(-3R^2, \frac{\tau R^2}{\eta^2}\right]} |f|^{n+1} \right)^{\frac{1}{n+1}} \leq \frac{\varepsilon_1}{4} =: \varepsilon$$

for a uniform constant $0 < \varepsilon_1 < 1$ as in Lemma 6.1.

Then there exist constants $\sigma > 0$ and $\tilde{M}_0 > 1$ depending on n, λ, Λ and a_L such that for $\nu := \frac{\tilde{M}_0}{\tilde{M}_0 - 1/2} > 1$, the following holds:

If $j \geq 1$ is an integer and $z_1 \in M$ and $t_1 \in \mathbb{R}$ satisfy

$$d(z_o, z_1) \leq \kappa R, \quad |t_1| \leq \kappa^2 R^2$$

and

$$u(z_1, t_1) \geq \nu^{j-1} \tilde{M}_0,$$

then

- (i) $K_{\frac{50}{\eta^3}L_j, \left(3 + \frac{\tau}{\eta^2}\right)L_j^2}(z_1, t_1) \subset B_{\frac{50}{\eta^3}R}(z_o) \times \left(-3R^2, \frac{\tau R^2}{\eta^2}\right]$,
- (ii) $\sup_{K_{\frac{50}{\eta^3}L_j, \left(3 + \frac{\tau}{\eta^2}\right)L_j^2}(z_1, t_1)} u \geq \nu^j \tilde{M}_0$,

where $L_j := \sigma \tilde{M}_0^{-\frac{\epsilon}{n+2}} \nu^{-\frac{j\epsilon}{n+2}} R$ and $0 < \epsilon < 1$ as in Corollary 6.2.

Proof. We select $\sigma > 0$ and $\tilde{M}_0 > 1$ large so that

$$\sigma > \frac{c_2}{c_1 \delta_0} (2d2^\epsilon)^{\frac{1}{n+2}}$$

and

$$\sigma \tilde{M}_0^{-\frac{\epsilon}{n+2}} + d \tilde{M}_0^{-\epsilon} \leq \frac{\kappa}{4},$$

where d, ϵ, c_1, c_2 and δ_0 are the constants in Corollary 6.2 and Theorem 5.1. Since $L_j \leq \frac{\kappa R}{4} < \frac{\eta R}{8}$, $d(z_o, z_1) \leq \kappa R < R$ and $|t_1| \leq \kappa^2 R^2 < \frac{\eta^2 R^2}{4}$, we have

$$B_{\frac{50}{\eta^3}L_j}(z_1) \times \left(t_1 - \left(3 + \frac{\tau}{\eta^2}\right)L_j^2, t_1\right] \subset B_{\frac{50}{\eta^3}R}(z_o) \times \left(-3R^2, \frac{\tau R^2}{\eta^2}\right],$$

so (i) is true.

Now, suppose on the contrary that

$$\sup_{K_{\frac{50}{\eta^3}L_j, \left(3 + \frac{\tau}{\eta^2}\right)L_j^2}(z_1, t_1)} u < \nu^j \tilde{M}_0.$$

Let $k_j := k_{L_j} \geq k_R$ with L_j in Definition 5.2. From Lemma 5.5, there exists a dyadic cube Q_{L_j} of generation k_j such that $d(z_1, Q_{L_j}) \leq \delta_1 L_j$. We also find a parabolic dyadic cube K_{L_j} of generation k_j such that

$$K_{L_j} \subset Q_{L_j} \times \left(t_1 - \frac{\tau L_j^2}{\eta^2} - L_j^2, t_1 - \frac{\tau L_j^2}{\eta^2} + L_j^2\right)$$

since $l(k_j) < \delta_0^2 L_j^2$. Let K_1 be the unique predecessor of K_{L_j} of generation k_R , that is,

$$K_{L_j} \subset K_1 := Q_1 \times (a - l(k_R), a].$$

Then we have

$$d(z_o, Q_1) \leq d(z_o, Q_{L_j}) \leq d(z_o, z_1) + d(z_1, Q_{L_j}) \leq \kappa R + \delta_1 L_j < \delta_1 R$$

$$\text{and } (a - l(k_R), a] \subset (-R^2, R^2)$$

since

$$l(k_R) + |t_1| + \frac{\tau L_j^2}{\eta^2} + L_j^2 \leq l(k_R) + |t_1| + \frac{16L_j^2}{\eta^2} + L_j^2$$

$$\leq \delta_0^2 R^2 + \kappa^2 R^2 + \left(\frac{16}{\eta^2} + 1\right) \frac{\kappa^2}{16} R^2 = \left\{ \delta_0^2 + \left(\frac{16}{\eta^2} + 17\right) \frac{\eta^4(1 - \delta_0^2)}{64} \right\} R^2 < R^2.$$

Now, we apply Corollary 6.2 to u with K_1 to obtain

$$(25) \quad \left| \left\{ u \geq \nu^j \frac{\tilde{M}_0}{2} \right\} \cap K_{L_j} \right| \leq \left| \left\{ u \geq \nu^j \frac{\tilde{M}_0}{2} \right\} \cap K_1 \right| \leq d \left(\nu^j \frac{\tilde{M}_0}{2} \right)^{-\epsilon} |K_1|.$$

On the other hand, we consider the function

$$w := \frac{\nu \tilde{M}_0 - u / \nu^{j-1}}{(\nu - 1) \tilde{M}_0},$$

which is nonnegative and satisfies

$$\mathcal{L}w = -\frac{f}{\nu^{j-1}(\nu - 1)\tilde{M}_0} \quad \text{in } K_{\frac{50}{\eta^3}L_j, \left(3 + \frac{\tau}{\eta^2}\right)L_j^2}(z_1, t_1)$$

from the assumption. We also have $w(z_1, t_1) \leq 1$ and

$$\frac{|f|}{\nu^{j-1}(\nu - 1)\tilde{M}_0} \leq \frac{|f|}{(\nu - 1)\tilde{M}_0} = \frac{2(\tilde{M}_0 - 1/2)|f|}{\tilde{M}_0} \leq 2|f|.$$

By using the volume comparison theorem with $L_j \leq \frac{\kappa}{4}R < \frac{\eta R}{8}$ and $B_{\frac{11}{\eta} \frac{4}{\eta^2} \frac{\eta R}{8}}(z_o) \subset B_{\frac{50}{\eta^3} \frac{\eta R}{8}}(z_1)$, we get

$$L_j^2 \left(\int_{K_{\frac{50}{\eta^3}L_j, \left(3 + \frac{\tau}{\eta^2}\right)L_j^2}(z_1, t_1)} |2f|^{n+1} \right)^{\frac{1}{n+1}} = \frac{2L_j^2}{|B_{\frac{50}{\eta^3}L_j}(z_1)|^{\frac{1}{n+1}} \left\{ \left(3 + \frac{\tau}{\eta^2}\right)L_j^2 \right\}^{\frac{1}{n+1}}} \|f\|_{L^{n+1}}$$

$$\leq \frac{2(\eta R/8)^2}{|B_{\frac{50}{\eta^3} \frac{\eta R}{8}}(z_1)|^{\frac{1}{n+1}} \left\{ \left(3 + \frac{\tau}{\eta^2}\right)(\eta R/8)^2 \right\}^{\frac{1}{n+1}}} \|f\|_{L^{n+1}}$$

$$\leq \frac{2(\eta R/8)^2}{|B_{\frac{11}{\eta} \frac{4}{\eta^2} \frac{\eta R}{8}}(z_o)|^{\frac{1}{n+1}} \left\{ \left(3 + \frac{\tau}{\eta^2}\right)(\eta R/8)^2 \right\}^{\frac{1}{n+1}}} \|f\|_{L^{n+1}}$$

$$\leq \frac{2R^2}{|B_{\frac{11}{\eta} \frac{4}{\eta^2} R}(z_o)|^{\frac{1}{n+1}} \left\{ \left(3 + \frac{\tau}{\eta^2}\right)R^2 \right\}^{\frac{1}{n+1}}} \|f\|_{L^{n+1} \left(B_{\frac{50}{\eta^3}R}(z_o) \times \left(-3R^2, \frac{\tau R^2}{\eta^2}\right] \right)}$$

$$\leq 2 \left(\frac{50}{44} \right)^{\frac{n}{n+1}} \frac{\varepsilon_1}{4} \leq \varepsilon_1.$$

Applying Corollary 6.2 to w in K_{L_j} , we deduce that $|\{w \geq \tilde{M}_0\} \cap K_{L_j}| \leq d\tilde{M}_0^{-\epsilon}|K_{L_j}|$, i.e.,

$$\left| \left\{ u \leq \nu^j \frac{\tilde{M}_0}{2} \right\} \cap K_{L_j} \right| \leq d\tilde{M}_0^{-\epsilon}|K_{L_j}|.$$

Putting together with (25), we obtain

$$|K_{L_j}| \leq 2d2^\epsilon v^{-j\epsilon} \tilde{M}_0^{-\epsilon} |K_1|$$

since $d\tilde{M}_0^{-\epsilon} \leq \frac{\kappa}{2} < 1/2$. From Theorem 5.1, there is a point $z_* \in Q_{L_j}$ such that $B_{c_1\delta_0^{k_j}}(z_*) \subset Q_{L_j} \subset Q_1 \subset \bar{B}_{c_2\delta_0^{k_R}}(z_*)$. Then we have

$$\begin{aligned} \left| B_{c_1\delta_0^{k_j}}(z_*) \right| \cdot c_1^2 \delta_0^{2k_j} &\leq \left| B_{c_1\delta_0^{k_j}}(z_*) \right| \cdot l(k_j) \leq |K_{L_j}| \\ &\leq 2d2^\epsilon v^{-j\epsilon} \tilde{M}_0^{-\epsilon} |K_1| = 2d2^\epsilon v^{-j\epsilon} \tilde{M}_0^{-\epsilon} |Q_1| \cdot l(k_R) \\ &< 2d2^\epsilon v^{-j\epsilon} \tilde{M}_0^{-\epsilon} |\bar{B}_{c_2\delta_0^{k_R}}(z_*)| \cdot c_2^2 \delta_0^{2k_R} \\ &\leq 2d2^\epsilon v^{-j\epsilon} \tilde{M}_0^{-\epsilon} \left(\frac{c_2 \delta_0^{k_R}}{c_1 \delta_0^{k_j}} \right)^n \left| B_{c_1\delta_0^{k_j}}(z_*) \right| c_2^2 \delta_0^{2k_R} \end{aligned}$$

from the volume comparison theorem. This means

$$\delta_0^{k_j} < (2d2^\epsilon)^{\frac{1}{n+2}} \tilde{M}_0^{-\frac{\epsilon}{n+2}} v^{-\frac{j\epsilon}{n+2}} \frac{c_2}{c_1} \delta_0^{k_R}.$$

Since $c_2 \delta_0^{k_R-1} < R \leq c_2 \delta_0^{k_R-2}$, we deduce that

$$\begin{aligned} L_j &\leq c_2 \delta_0^{k_j-2} \leq \frac{c_2^2}{c_1 \delta_0^2} (2d2^\epsilon)^{\frac{1}{n+2}} \tilde{M}_0^{-\frac{\epsilon}{n+2}} v^{-\frac{j\epsilon}{n+2}} \delta_0^{k_R} \\ &< \frac{c_2}{c_1 \delta_0} (2d2^\epsilon)^{\frac{1}{n+2}} \tilde{M}_0^{-\frac{\epsilon}{n+2}} v^{-\frac{j\epsilon}{n+2}} R < \sigma \tilde{M}_0^{-\frac{\epsilon}{n+2}} v^{-\frac{j\epsilon}{n+2}} R = L_j, \end{aligned}$$

in contradiction to the definition of L_j . Therefore, (ii) is true. \square

Thus we deduce the following lemma from Lemma 6.4.

Lemma 6.5. *Suppose that M satisfies the conditions (3),(4). Let $z_o \in M, R > 0$ and $\tau \in [3, 16]$. Let u be a nonnegative smooth function such that $\mathcal{L}u = f$ in $B_{\frac{50}{3}R}(z_o) \times (-3R^2, \frac{\tau R^2}{\eta^2}]$. Assume that*

$$\inf_{B_R(z_o) \times \left[\frac{2R^2}{\eta^2}, \frac{\tau R^2}{\eta^2} \right]} u \leq 1$$

and

$$R^2 \left(\int_{B_{\frac{50}{3}R}(z_o) \times (-3R^2, \frac{\tau R^2}{\eta^2}]} |f|^{n+1} \right)^{\frac{1}{n+1}} \leq \varepsilon$$

for a uniform constant $0 < \varepsilon < 1$ in Lemma 6.4. Then

$$\sup_{B_{\frac{\kappa R}{2}}(z_o) \times \left(-\frac{\kappa^2 R^2}{4}, \frac{\kappa^2 R^2}{4} \right)} u \leq C,$$

where $C > 0$ depends only on n, λ, Λ and a_L .

Proof. We take $j_o \in \mathbb{N}$ such that

$$\sum_{j=j_o}^{\infty} \frac{50}{\eta^3} L_j < \frac{\kappa R}{2} \quad \text{and} \quad \sum_{j=j_o}^{\infty} \left(3 + \frac{16}{\eta^2} \right) L_j^2 < \frac{\kappa^2 R^2}{4}.$$

We claim that $\sup_{B_{\frac{\kappa R}{2}}(z_o) \times \left(-\frac{\kappa^2 R^2}{4}, \frac{\kappa^2 R^2}{4}\right)} u \leq \nu^{j_o-1} \tilde{M}_0$ with $\tilde{M}_0 > 1$ as in Lemma 6.4. If it does

not hold, then there is a point $(z_{j_o}, t_{j_o}) \in B_{\frac{\kappa R}{2}}(z_o) \times \left(-\frac{\kappa^2 R^2}{4}, \frac{\kappa^2 R^2}{4}\right)$ such that $u(z_{j_o}, t_{j_o}) > \nu^{j_o-1} \tilde{M}_0$. Applying Lemma 6.4 with $(z_1, t_1) = (z_{j_o}, t_{j_o})$, we can find a point $(z_{j_o+1}, t_{j_o+1}) \in K_{\frac{50}{\eta^3} L_j, \left(3 + \frac{\tau}{\eta^2}\right) L_j^2}(z_{j_o}, t_{j_o})$ such that

$$u(z_{j_o+1}, t_{j_o+1}) \geq \nu^{j_o} \tilde{M}_0.$$

According to the choice of j_o , we have

$$d(z_o, z_{j_o+1}) \leq d(z_o, z_{j_o}) + d(z_{j_o}, z_{j_o+1}) < \frac{\kappa R}{2} + \frac{\kappa R}{2} = \kappa R$$

and

$$|t_{j_o+1}| \leq |t_{j_o}| + |t_{j_o} - t_{j_o+1}| < \frac{\kappa^2 R^2}{4} + \frac{\kappa^2 R^2}{4} < \kappa^2 R^2.$$

Thus we iterate this argument to obtain a sequence of points (z_j, t_j) for $j \geq j_o$ satisfying

$$d(z_o, z_j) \leq \kappa R, \quad |t_j| \leq \kappa^2 R^2 \quad \text{and} \quad u(z_j, t_j) \geq \nu^{j-1} \tilde{M}_0,$$

since $d(z_o, z_j) \leq d(z_o, z_{j_o}) + \sum_{i=j_o}^{\infty} d(z_i, z_{i+1}) \leq \frac{\kappa R}{2} + \sum_{i=j_o}^{\infty} \frac{50}{\eta^3} L_i < \kappa R$ and $|t_i| \leq |t_{j_o}| + \sum_{i=j_o}^{\infty} |t_i - t_{i+1}| \leq$

$\frac{\kappa^2 R^2}{4} + \sum_{i=j_o}^{\infty} \left(3 + \frac{\tau}{\eta^2}\right) L_i^2 < \kappa^2 R^2$ for $j \geq j_o$. This contradicts to the continuity of u and therefore we conclude that

$$\sup_{B_{\frac{\kappa R}{2}}(z_o) \times \left(-\frac{\kappa^2 R^2}{4}, \frac{\kappa^2 R^2}{4}\right)} u \leq \nu^{j_o-1} \tilde{M}_0.$$

□

Now the Harnack inequality in Theorem 6.6 follows easily from Lemma 6.5 by using a standard covering argument and the volume comparison theorem.

Theorem 6.6 (Harnack Inequality). *Suppose that M satisfies the conditions (3),(4). Let $z_o \in M$, and $R > 0$. Let u be a nonnegative smooth function in $K_{2R}(0, 4R^2) \subset M \times \mathbb{R}$. Then*

$$(26) \quad \sup_{K_R(z_o, 2R^2)} u \leq C \left\{ \inf_{K_R(z_o, 4R^2)} u + R^2 \left(\int_{K_{2R}(z_o, 4R^2)} |\mathcal{L}u|^{n+1} \right)^{\frac{1}{n+1}} \right\},$$

where $C > 0$ is a constant depending only on n, λ, Λ and a_L .

Proof. According to Lemma 6.5, for $\tau \in [3, 16]$, a nonnegative smooth function v in $K_{\frac{50}{\eta^3} r, \left(3 + \frac{\tau}{\eta^2}\right) r^2}(\bar{x}, \bar{t} + \frac{\tau r^2}{\eta^2})$ satisfies

$$(27) \quad \sup_{K_{\frac{\kappa R}{2}}(\bar{x}, \bar{t})} v \leq C \left\{ \inf_{K_{\frac{\kappa R}{2}}(\bar{x}, \bar{t} + \frac{\tau r^2}{\eta^2})} v + r^2 \left(\int_{K_{\frac{50}{\eta^3} r, \left(3 + \frac{\tau}{\eta^2}\right) r^2}(\bar{x}, \bar{t} + \frac{\tau r^2}{\eta^2})} |\mathcal{L}v|^{n+1} \right)^{\frac{1}{n+1}} \right\}$$

since $\frac{\kappa}{2} < 1$.

Now, let $(x, t) \in K_R(z_o, 2R^2) = B_R(z_o) \times (R^2, 2R^2]$ and $(y, s) \in K_R(z_o, 4R^2) = B_R(z_o) \times (3R^2, 4R^2]$. We show that

$$u(x, t) \leq C \left\{ u(y, s) + \frac{R^2}{|K_{2R}(z_o, 4R^2)|^{\frac{1}{n+1}}} \|\mathcal{L}u\|_{L^{n+1}(K_{2R}(z_o, 4R^2))} \right\}$$

for a uniform constant $C > 0$ depending only on n, λ, Λ and a_L . We consider a piecewise C^1 path $\gamma : [0, l] \rightarrow M$, $\gamma(0) = x, \gamma(l) = y$, $l < 2R$, consisting of a minimal geodesic parametrized by arc length joining x and z_o , followed by a minimal geodesic parametrized by arc length joining z_o and y . We notice that $\gamma([0, l]) \subset B_R(z_o)$ and $d(\gamma(s_1), \gamma(s_2)) \leq |s_1 - s_2|$.

We can select uniform constants $A > 0$ and $N \in \mathbb{N}$ such that

$$A := \max \left\{ \frac{64}{3\kappa\eta^2}, \frac{50}{\eta^3} \right\} \quad \text{and} \quad \frac{3}{16}\eta^2 A^2 \leq N \leq \frac{1}{3}\eta^2 A^2$$

since $\frac{16-9}{16 \cdot 3}\eta^2 A^2 \geq \frac{7 \cdot 4}{3^2 \kappa} A > 1$. For $i = 0, 1, \dots, N$, we define

$$(x_i, t_i) := \left(\gamma\left(i \frac{l}{N}\right), i \frac{s-t}{N} + t \right) \in B_R(z_o) \times [R^2, 4R^2].$$

Then we have $(x_0, t_0) = (x, t)$, $(x_N, t_N) = (y, s)$ and for $i = 0, \dots, N-1$,

$$\begin{aligned} d(x_{i+1}, x_i) &\leq \frac{l}{N} < \frac{2R}{N} \leq \frac{64}{3\kappa\eta^2 A} \cdot \frac{\kappa R}{2A} \leq \frac{\kappa R}{2A}, \\ \frac{3R^2}{\eta^2 A^2} &\leq \frac{R^2}{N} \leq t_{i+1} - t_i = \frac{s-t}{N} \leq \frac{3R^2}{N} \leq \frac{16R^2}{\eta^2 A^2}. \end{aligned}$$

We also have that $K_{\frac{50}{\eta^3} \frac{R}{A}, 3 \frac{R^2}{A^2} + t_i - t_{i-1}}(x_i, t_i) \subset K_{2R}(z_o, 4R^2)$ for $i = 1, \dots, N$ since $\frac{50}{\eta^3} \frac{R}{A} \leq R$.

We apply the estimate (27) with $r = \frac{R}{A}$, $\tau = (t_{i+1} - t_i) \frac{\eta^2 A^2}{R^2}$ and $(\bar{x}, \bar{t}) = (x_{i+1}, t_{i+1})$ for $i = 0, 1, \dots, N-1$ and use the volume comparison theorem to have

$$\begin{aligned} u(x_i, t_i) &\leq C \left\{ u(x_{i+1}, t_{i+1}) + \frac{(R/A)^2}{|K_{\frac{50}{\eta^3} \frac{R}{A}, 3 \frac{R^2}{A^2} + t_i - t_{i-1}}(x_{i+1}, t_{i+1})|^{\frac{1}{n+1}}} \|\mathcal{L}u\|_{L^{n+1}(K_{2R}(z_o, 4R^2))} \right\} \\ &\leq C \left\{ u(x_{i+1}, t_{i+1}) + \frac{(R/A)^2}{|B_{\frac{50}{\eta^3} \frac{R}{A}}(x_{i+1}) \cdot \left(3 + \frac{3}{\eta^2}\right) \frac{R^2}{A^2}|^{\frac{1}{n+1}}} \|\mathcal{L}u\|_{L^{n+1}(K_{2R}(z_o, 4R^2))} \right\} \\ &\leq C \left\{ u(x_{i+1}, t_{i+1}) + \frac{R^2}{|B_{3R}(x_{i+1}) \cdot 4R^2|^{\frac{1}{n+1}}} \|\mathcal{L}u\|_{L^{n+1}(K_{2R}(z_o, 4R^2))} \right\}, \end{aligned}$$

where a uniform constant $C > 0$ may change from line to line. Since $B_{3R}(x_{i+1}) \supset B_{2R}(z_o)$, we deduce that

$$u(x_i, t_i) \leq C \left\{ u(x_{i+1}, t_{i+1}) + \frac{R^2}{|B_{2R}(z_o) \cdot 4R^2|^{\frac{1}{n+1}}} \|\mathcal{L}u\|_{L^{n+1}(K_{2R}(z_o, 4R^2))} \right\}.$$

Therefore, we conclude that

$$u(x, t) \leq C \left\{ u(y, s) + \frac{R^2}{|K_{2R}(z_o, 4R^2)|^{\frac{1}{n+1}}} \|\mathcal{L}u\|_{L^{n+1}(K_{2R}(z_o, 4R^2))} \right\}$$

for a uniform constant $C > 0$ since $N \in \mathbb{N}$ is uniform. \square

Arguing in a similar way as Theorem 6.6, Corollary 6.3 gives the following weak Harnack inequality.

Theorem 6.7 (Weak Harnack Inequality). *Suppose that M satisfies the conditions (3),(4). Let $z_o \in M$, and $R > 0$. Let u be a nonnegative smooth function such that $\mathcal{L}u \leq f$ in*

$K_{2R}(z_o, 4R^2)$. Then

$$\left(\int_{K_R(z_o, 2R^2)} u^{p_o} \right)^{\frac{1}{p_o}} \leq C \left\{ \inf_{K_R(z_o, 4R^2)} u + R^2 \left(\int_{K_{2R}(z_o, 4R^2)} |f^{c+}|^{n+1} \right)^{\frac{1}{n+1}} \right\},$$

where $0 < p_o < 1$ and $C > 0$ depend only on n, λ, Λ and a_L .

Proof. Let $\epsilon > 0$ be the constant in Corollary 6.2 and let $p_o := \frac{\epsilon}{2}$. We consider a parabolic decomposition of $M \times (0, 4R^2]$ according to Lemma 5.3. Let $k := k_{\frac{\kappa R}{A}}$ for the constant $A > 0$ in the proof of Theorem 6.6. Let $\{K^{k, \alpha} := Q^{k, \alpha} \times (t^{k, \alpha} - l(k), t^{k, \alpha}]\}_{\alpha \in J'_k}$ be a family of parabolic dyadic cubes intersecting $K_R(z_o, 2R^2)$. We note that $\text{diam}(Q^{k, \alpha}) \leq c_2 \delta_0^k \leq \delta_0 \cdot \frac{\kappa R}{A}$ and $l(k) \leq \delta_0^2 \cdot \frac{\kappa^2 R^2}{A^2}$. Following the same argument as Corollary 6.3, we deduce that $|J'_k|$ is uniformly bounded and

$$\int_{K_R(z_o, 2R^2)} u^{p_o} \leq |J'_k| \int_{K^{k, \alpha}} u^{p_o}$$

for some $K^{k, \alpha}$ with $\alpha \in J'_k$. Then we find $(x, t) \in K^{k, \alpha} \cap B_R(z_o) \times [R^2, (2 + \delta_0^2 \frac{\kappa^2}{A^2})R^2]$ such that $K^{k, \alpha} \subset K_{\frac{\kappa R}{A}}(x, t)$ since $\text{diam}(Q^{k, \alpha}) \leq \delta_0 \cdot \frac{\kappa R}{A}$ and $l(k) \leq \delta_0^2 \cdot \frac{\kappa^2 R^2}{A^2}$. Since $d(z_o, x) \leq R$ and $B_{\frac{\kappa R}{A}}(x) \subset B_{(1 + \frac{\kappa}{A})R}(z_o)$, we have

$$(28) \quad \frac{1}{|K_R(z_o, 2R^2)|} \int_{K_R(z_o, 2R^2)} u^{p_o} \leq \frac{C_0}{|K_{\frac{\kappa R}{A}}(x, t)|} \int_{K_{\frac{\kappa R}{A}}(x, t)} u^{p_o}$$

for $C_0 := |J'_k| \left(1 + \frac{\kappa}{A}\right)^n \cdot \frac{\kappa^2}{A^2}$ by using the volume comparison theorem.

We set

$$\inf_{K_R(z_o, 4R^2)} u =: u(y, s)$$

for some $(y, s) \in \overline{K_R(z_o, 4R^2)}$. As in the proof of Theorem 6.6 we take a piecewise geodesic path γ connecting x to y . Let $N \in \mathbb{N}$ be the constant in Theorem 6.6. For $i = 0, 1, \dots, N$, we define

$$(x_i, t_i) := \left(\gamma \left(i \frac{l}{N} \right), i \frac{s-t}{N} + t \right) \in B_R(z_o) \times [R^2, 4R^2].$$

Then we have $(x_0, t_0) = (x, t)$, $(x_N, t_N) = (y, s)$ and for $i = 0, \dots, N-1$,

$$d(x_{i+1}, x_i) < \frac{\kappa}{2} \cdot \frac{R}{A} \quad \text{and} \quad \frac{3R^2}{\eta^2 A^2} \leq t_{i+1} - t_i \leq \frac{16R^2}{\eta^2 A^2}.$$

It is easy to check that for any $i = 0, 1, \dots, N-1$, $B_{\frac{\kappa R}{A}}(x_i) \cap B_{\frac{\kappa R}{A}}(x_{i+1}) \supset B_{\frac{\kappa R}{2A}}(x_{i+1})$ and hence

$$(29) \quad \begin{aligned} \inf_{K_{\frac{\kappa R}{A}}(x_i, t_{i+1})} u &\leq \inf_{K_{\frac{\kappa R}{2A}}(x_{i+1}, t_{i+1})} u \leq \left\{ \frac{1}{|K_{\frac{\kappa R}{2A}}(x_{i+1}, t_{i+1})|} \int_{K_{\frac{\kappa R}{2A}}(x_{i+1}, t_{i+1})} u^{p_o} \right\}^{\frac{1}{p_o}} \\ &\leq 2^{\frac{n+2}{p_o}} \left\{ \frac{1}{|K_{\frac{\kappa R}{A}}(x_{i+1}, t_{i+1})|} \int_{K_{\frac{\kappa R}{A}}(x_{i+1}, t_{i+1})} u^{p_o} \right\}^{\frac{1}{p_o}}. \end{aligned}$$

On the other hand, Corollary 6.3 says that for $i = 0, 1, \dots, N-1$

$$\begin{aligned} & \left\{ \frac{1}{|K_{\frac{R}{A}}(x_i, t_i)|} \int_{K_{\frac{R}{A}}(x_i, t_i)} u^{p_o} \right\}^{1/p_o} \\ & \leq C \left\{ \inf_{K_{\frac{R}{A}}(x_i, t_{i+1})} u + \frac{(R/A)^2}{\left| K_{\frac{50}{\eta^3} \frac{R}{A}, 3 \frac{R^2}{A^2} + t_{i+1} - t_i}(x_i, t_{i+1}) \right|^{\frac{1}{n+1}}} \|f^+\|_{L^{n+1}(K_{2R}(z_o, 4R^2))} \right\} \\ & \leq C \left\{ \inf_{K_{\frac{R}{A}}(x_i, t_{i+1})} u + \frac{R^2}{|K_{2R}(z_o, 4R^2)|^{\frac{1}{n+1}}} \|f^+\|_{L^{n+1}(K_{2R}(z_o, 4R^2))} \right\} \end{aligned}$$

by using the same argument as Theorem 6.6 with the volume comparison theorem. Combining with (29), we deduce

$$\begin{aligned} & \left\{ \frac{1}{|K_{\frac{R}{A}}(x, t)|} \int_{K_{\frac{R}{A}}(x, t)} u^{p_o} \right\}^{1/p_o} \\ & \leq C \left\{ \inf_{K_{\frac{R}{A}}(x_{N-1}, t_N)} u + \frac{R^2}{|K_{2R}(z_o, 4R^2)|^{\frac{1}{n+1}}} \|f^+\|_{L^{n+1}(K_{2R}(z_o, 4R^2))} \right\} \\ & \leq C \left\{ \inf_{K_{\frac{R}{2A}}(x_N, t_N)} u + \frac{R^2}{|K_{2R}(z_o, 4R^2)|^{\frac{1}{n+1}}} \|f^+\|_{L^{n+1}(K_{2R}(z_o, 4R^2))} \right\} \\ & \leq C \left\{ u(y, s) + \frac{R^2}{|K_{2R}(z_o, 4R^2)|^{\frac{1}{n+1}}} \|f^+\|_{L^{n+1}(K_{2R}(z_o, 4R^2))} \right\}, \end{aligned}$$

for a uniform constant $C > 0$ since $N \in \mathbb{N}$ is uniform. Therefore, the result follows from (28). \square

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SEICK KIM:, DEPARTMENT OF COMPUTATIONAL SCIENCE AND ENGINEERING, YONSEI UNIVERSITY, 50 YONSEI-RO, SEODAEMUN-GU, SEOUL, 120-749, REPUBLIC OF KOREA

E-mail address: kimseick@yonsei.ac.kr

SOOJUNG KIM:, SCHOOL OF MATHEMATICAL SCIENCES, SEOUL NATIONAL UNIVERSITY, 1 GWANAK-RO, GWANAK-GU, SEOUL, 151-747, REPUBLIC OF KOREA

E-mail address: soojung26@gmail.com

KI-AHM LEE:, SCHOOL OF MATHEMATICAL SCIENCES, SEOUL NATIONAL UNIVERSITY, 1 GWANAK-RO, GWANAK-GU, SEOUL, 151-747, REPUBLIC OF KOREA

E-mail address: kiahm@math.snu.ac.kr